

Name: _____

Student ID#: _____

Section: _____

Final Exam

Wednesday March 20

MAT 21D, Temple, Winter 2013

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (25pts): (a) Sketch the region of integration \mathbf{R}_{xy} determined by the iterated integral

$$\int_0^1 \int_{x^2}^x xy^2 dy dx. \quad (1)$$

and evaluate it.

(b) Rewrite (1) with order of integration reversed, and evaluate it.

Problem #2 (25pts): A particle moves along a trajectory given by

$$\mathbf{r}(t) = 2 \sin t \, \mathbf{i} + 2 \cos t \, \mathbf{j} - t \, \mathbf{k}.$$

Find the following:

(a) The velocity vector $\mathbf{v}(t)$

(b) The speed $v(t)$

(c) The unit tangent vector $\mathbf{T}(t)$

(d) The unit normal $\mathbf{N}(t)$

(e) Find the acceleration vector $\mathbf{a}(t)$

(f) Find a_T and a_N such that $\mathbf{a}(t) = a_T \mathbf{T} + a_N \mathbf{N}$.

(g) Find the arclength from $t = 0$ to $t = 1$.

Problem #3 (25pts): Let $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

(a) Derive the formula $\frac{\partial}{\partial x}r(x, y, z) = \frac{x}{r}$.

(b) Write the corresponding formula for y and z .

(c) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $\mathbf{F} = -\frac{\mathbf{r}}{r^3}$ be the gravitational force field. Use (a) to show that $\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3}$, (i.e., \mathbf{F} is conservative.)

(d) Let \mathcal{C} be any curve taking $A = (1, -1, 2)$ to $B = (-3, 0, 4)$. Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \, ds$, (the work done by F along \mathcal{C}).

Problem #4 (25pts): (a) Use the Leibniz substitution principle to show that in general, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a parameterization of curve C for $a \leq t \leq b$, and $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field, then the following are equal:

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{v} \, dt = \int_C Mdx + Ndy + Pdz.$$

#4 (b) Assume further that $\mathbf{r}(t)$, $a \leq t \leq b$ is a parameterization of \mathcal{C} which describes the motion of an object subject to the total force $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a} = \mathbf{r}''(\mathbf{t})$, and $v(t) = \|\mathbf{r}'(\mathbf{t})\|$ is the velocity. Prove that

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \frac{1}{2}mv^2(b) - \frac{1}{2}mv^2(a).$$

Problem #5 (25pts): Let $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z)\mathbf{k}$. Find:

(a) Curl \mathbf{F}

(c) Find the circulation per area in \mathbf{F} around an axis aligned with the direction $\mathbf{v} = \mathbf{i} + \mathbf{j}$ at the point $(1, -2, -1)$.

Problem #6 (25pts): Let \mathcal{S} be the surface in \mathcal{R}^3 which is the image of $0 \leq u \leq 1$, $0 \leq v \leq 2$ under the parameterization $\mathbf{r}(u, v) = u^2\mathbf{i} + v^2\mathbf{j} - 2uv\mathbf{k}$. Evaluate the flux integral

$$\int \int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, d\sigma,$$

where \mathbf{F} is the vector field $\mathbf{F} = z\mathbf{i}$, and \mathbf{n} points toward negative x .

Problem #7 (25pts): Verify Stokes Theorem:

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds = \int \int_{\mathcal{S}} \text{Curl} \mathbf{F} \cdot \mathbf{n} d\sigma,$$

where \mathcal{S} is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, and \mathcal{C} is the closed curve $x^2 + y^2 = 4$.

Problem #8 (25pts): Consider a fluid flow with $\delta(x, y, z)$ the density in $\frac{kg}{m^3}$, $\mathbf{v}(x, y, z)$ the velocity in $\frac{m}{s}$, and let $\mathbf{F} = \delta\mathbf{v}$ be the mass flux vector, given by $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + (x^2 + y^2)\mathbf{k}$.

(a) Find $\text{Div } \mathbf{F}$.

(c) Find the rate at which mass is passing upward through the disk $x^2 + y^2 \leq 1$, $z = 0$.