FINAL EXAM
Math 21D
Temple-F06

–Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

–Show all of your work, and justify your answers for full credit.

<u>SCORES</u>

#1

#2

#3

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#9

TOTAL:

Problem 1. (22pts) Evaluate $\int\int_R x^2 y \, dA$ where R is the region in the xy-plane bounded between the curves $y = \sqrt{x}$ and $y = x$.

Problem 2. (22pts) Recall that $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \, ds = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{v} \, dt = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} = \int_{\mathcal{C}} M dx + N dy + P dz$ gives the line integral of a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ along curve $\mathcal{C}$. Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \, ds$ by parameterization, where $\mathcal{C}$ is the straight line from $P = (1, -1, 0)$ to $Q = (2, 0, 1)$, and $\mathbf{F} = y\mathbf{i} + 2x\mathbf{j} - z\mathbf{k}$.

Problem 3. (22pts) A parabolic mirror for a telescope occupies the surface $\mathcal{S}$ given by $z = x^2 + y^2$ for $-1 \le x \le 1$, $-1 \le y \le 1$, where x and y are in meters. The density of the mirror is $\delta(x, y, z) = \sqrt{1 + 4x^2 + 4y^2}$ kg/m^3. Find the mass of the mirror.

Problem 4. (22pts) Let $\mathcal{R}$ be the area in the xy-plane bounded by a simple closed curve $\mathcal{C}$. Green's theorem states that $\int_{\mathcal{C}} M dx + N dy = \int\int_{\mathcal{R}} N_x - M_y \, dx dy$. Use Green's theorem to derive *two different* line integrals over $\mathcal{C}$ that give the area of $\mathcal{R}$.

Problem 5. (22pts) Let $\mathbf{r} = (u + v)\mathbf{i} + (v^2 - u^2)\mathbf{j} + 3\mathbf{k}$ be the uv-parameterization of a surface $\mathcal{S}$, $0 \le u \le 1$, $0 \le v \le 1$, and let $\mathbf{F} = x\mathbf{k}$. Evaluate the flux $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, d\sigma$, where $\mathbf{n}$ is the upward pointing normal to $\mathbf{S}$. (Hint: $\mathbf{r}_u \times \mathbf{r}_v$ is perpendicular to $\mathcal{S}$.)

Problem 6. (22pts) Let $\int\int\int_D dV$ be given as an interated integral in the order $dxdzdy$ as

$$\int_0^2 \int_0^{6-3y} \int_0^{3-\frac{1}{2}z-\frac{3}{2}y} \, dxdzdy.$$

Sketch the region D in xyz-space, and set up the interated integral in the oder $\int\int\int \, dzdydx$. [DO NOT SOLVE]

Problem 7. (22pts) Let $\mathbf{x_0} = (x_0, y_0, z_0)$ be a point in 3-space, and let D_ϵ denote the disk with center $\mathbf{x_0}$ and radius ϵ, and let $\mathcal{C}_\epsilon$ denote the circle of radius ϵ around the boundary of the disk D_ϵ. Recall that Stokes Theorem says

$$\int_{\mathcal{C}_\epsilon} \mathbf{F} \cdot \mathbf{T} \, ds = \int\int_{D_\epsilon} \mathbf{Curl\ F} \cdot \mathbf{n} \, d\sigma,$$

where $\mathbf{T}$ and $\mathbf{n}$ are oriented by the right hand rule.

(a) In plain English say why $\int_{\mathcal{C}_\epsilon} \mathbf{F} \cdot \mathbf{T} \, ds$ is interperted as the circulation of $\mathbf{F}$ around the axis $\mathbf{n}$.

(b) Use Stokes Theorem together with (**a**) to derive the physical interpretation of $\mathbf{Curl\ F} \cdot \mathbf{n}$ evaluated at the point $\mathbf{x_0}$.

(c) Use (**a**) and (**b**) to show that **Curl F** evaluated at the point $\mathbf{x_0}$ points in the direction of the axis around which there is maximal circulation per area.

(d) Use (**c**) to show that the length of the **Curl F** is the magnitude of the circulation per area around the axis where it is maximal.

Problem 8. **(24pts)** Let $\mathbf{F}$ denote the vector field given by $\mathbf{F} = y\ \mathbf{i} + (x+2z)\ \mathbf{j} + (2y-1)\ \mathbf{k}$.

(a) Show that $Curl\mathbf{F} = \nabla \times \mathbf{F} = 0$, (implying that $\mathbf{F}$ is conservative).

(b) Find $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

(c) Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\ ds$ where $\mathcal{C}$ is line from $P = (1, 0, 2)$ to $Q = (0, 1, 0)$.

Problem 9. **(22pts)** Let $\delta = \delta(x, y, z, t)$ denote the density and $\mathbf{v} = \mathbf{v}(x, y, z, t)$ denote the velocity of a fluid moving through Euclidean three space, so that $\mathbf{F} = \delta\mathbf{v}$ is the mass flux vector. Recall the Divergence Theorem

$$\int\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\ d\sigma = \int\int\int_{D} Div\mathbf{F}\ dV,$$

which applies to any volume D with bounding surface $\mathcal{S}$, and $\mathbf{n}$ denotes the outward normal to $\mathcal{S}$.

(a) Give the physical meaning of both sides of the equation $\frac{d}{dt}\int\int\int_D \delta(x, y, z, t)\ dxdydz = -\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\ d\sigma$, and explain in words why this expresses the condition that mass is conserved in the volume D.

(b) Use **(a)** to derive the continuity equation $\delta_t + Div\ \delta\mathbf{v} = 0$, which expresses that mass is conserved everywhere in the flow.