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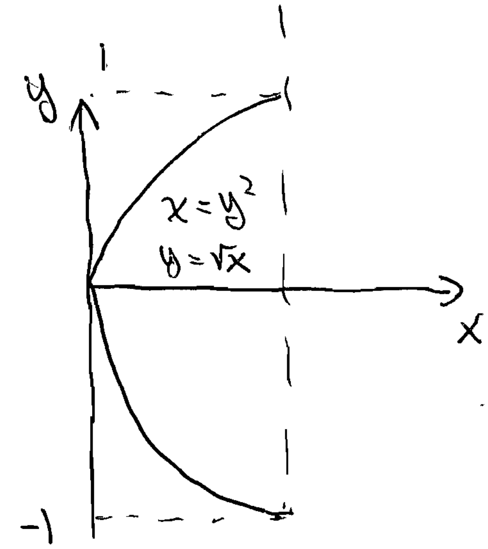
Final Exam Solutions
Math 21D F08

Problem #1



$$\int_{-1}^{1}\int_{x=y^2}^{x=1} \sqrt{x}\, y^2 \cos x^2 \, dx\, dy = \int_0^1 \int_{-\sqrt{x}}^{+\sqrt{x}} \sqrt{x}\, y^2 \cos x^2 \, dy\, dx$$

$$= \int_0^1 \sqrt{x} \cos x^2 \left. \frac{y^3}{3} \right]_{y=-\sqrt{x}}^{y=\sqrt{x}} dx$$

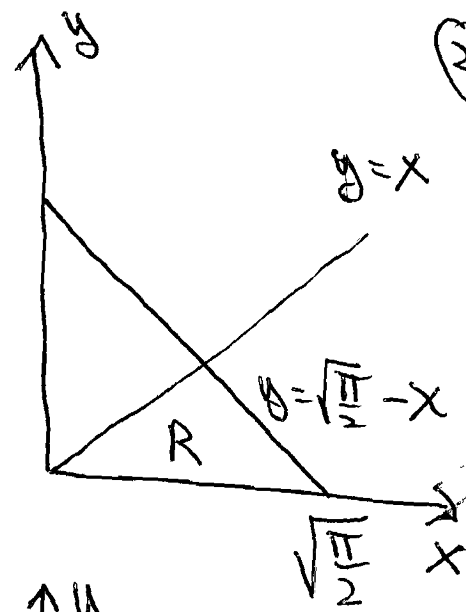
$$= \int_0^1 \sqrt{x} \cos x^2 \left(\frac{2\sqrt{x}^3}{3} \right) dx = \int_0^1 \frac{2x}{3} \cos x^2 \, dx = \frac{1}{3} \int_{x=0}^{x=1} \cos u \, du$$

$u = x^2$

$du = 2x\,dx$

$$= \frac{1}{3} \sin x^2 \Big]_{x=0}^{x=1} = \frac{1}{3} \sin(1)$$

Problem #2: $I = \iint_R \sin\{(x+y)^2\}\, dA$

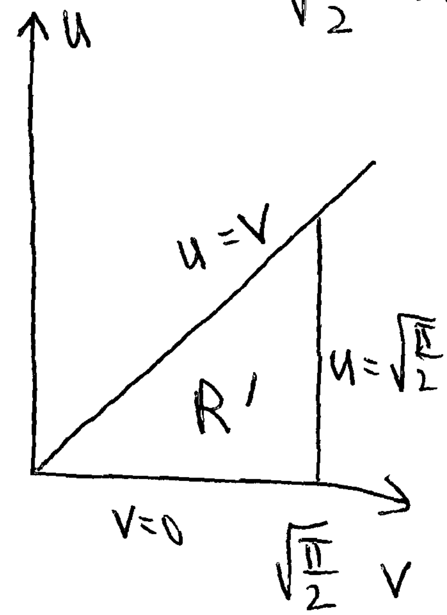


- $u = x+y \qquad x = u - y = u - \frac{v}{2}$

  $v = 2y \qquad y = \frac{v}{2}$

$y=0 \Rightarrow v=0$; $y = x \Rightarrow \frac{v}{2} = u - \frac{v}{2} \Rightarrow v = u$

$y = \sqrt{\frac{\pi}{2}} - x \Rightarrow \sqrt{\frac{\pi}{2}} - u + \frac{v}{2} = \frac{v}{2} \Rightarrow u = \sqrt{\frac{\pi}{2}}$



- $\vec{r}(u,v) = \left(u - \frac{v}{2}\right)\hat{i} + \frac{v}{2}\hat{j}$

$\vec{r}_u = \hat{i} \qquad \vec{r}_v = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}$

$$|\vec{r}_u \times \vec{r}_v| = \left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{matrix}\right| = \left|\frac{1}{2}\hat{k}\right| = \frac{1}{2}$$

$dx\,dy = |\vec{r}_u \times \vec{r}_v|\, du\, dv = \frac{1}{2}\, du\, dv$

- $I = \int_0^{\sqrt{\pi/2}} \int_0^{u} \sin u^2 \, dv\, du = \int_0^{\sqrt{\pi/2}} u \sin u^2 \, du$

$w = u^2$

$dw = 2u\,du$

$= -\frac{1}{2}\cos u^2 \Big]_0^{\sqrt{\pi/2}} = -\frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos 0$

$= \boxed{\frac{1}{2}}$

Problem #3.

$$\vec{F} = 2xyz\,\hat{i} + (x^2z + 2yz)\,\hat{j} + (x^2y + y^2 + 2z)\,\hat{k}$$

$$\frac{\partial f}{\partial x} = 2xyz \Rightarrow f = x^2yz + g(y,z)$$

$$\frac{\partial f}{\partial y} = x^2z + \frac{\partial g}{\partial y} = x^2z + 2yz \Rightarrow \frac{\partial g}{\partial y} = 2yz$$

$$\Rightarrow g = y^2z + h(z)$$

so $f = x^2yz + y^2z + h(z)$

$$\frac{\partial f}{\partial z} = x^2y + y^2 + h'(z) = x^2y + y^2 + 2z \Rightarrow h'(z) = 2z$$

$$= h(z) = z^2$$

$$\boxed{f(x,y,z) = x^2yz + y^2z + z^2}$$

Problem #4: $\vec{V} = x^2yz\,\hat{i} + (x^2+y^2)\,\hat{k}$

(a) $\text{Curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_t \\ x^2yz & 0 & x^2+y^2 \end{vmatrix} = \hat{i}(2y) - \hat{j}(2x - x^2y) + \hat{k}(-x^2z)$

(b) Axis $\parallel$ $\text{Curl}\,\vec{V}(1,1,-1) = 2\hat{i} - \hat{j} + \hat{k}$

(c) $\vec{N} = 2\hat{i} + \hat{k} \Rightarrow \vec{n} = \dfrac{2\hat{i} + \hat{k}}{\sqrt{2^2+1}} = \dfrac{1}{\sqrt{5}}\overrightarrow{(2,0,1)}$

Circl/area around $\vec{N}$ $= \text{Curl}\,\vec{V}(1,1,-1) \cdot \vec{n}$

$= \overrightarrow{(2,-1,1)} \cdot \dfrac{1}{\sqrt{5}}\overrightarrow{(2,0,1)} = \dfrac{1}{\sqrt{5}}(4 - 0 + 1)$

$= \boxed{\dfrac{5}{\sqrt{5}}} = \sqrt{5}$

Problem #5. $\vec{F} = x^2yz\,\hat{i} + (x^2+y^2)\,\hat{k}$ (5)

(a) $\text{Div}\,\vec{F} = (x^2yz)_x + (x^2+y^2)_z = 2xyz$

(b) $\dfrac{\text{Flux}}{\text{Vol}}$ at $(1,1,-1) = \text{Div}\,\vec{F}(1,1,-1) = 2(1\cdot 1\cdot(-1)) = -2$

(c) $I = \iint \vec{F}\cdot\vec{n}\,d\sigma$ = Flux of $\vec{F}$ upward thru

$\mathcal{S} \equiv x^2+y^2 \le 1$ $\qquad$ $\mathcal{S}$ ($\vec{n} = \hat{k}$ = upward normal)

= rate at which mass crosses upward thru $\mathcal{S}$

$$I = \iint_{x^2+y^2\le 1} \left((x^2yz), 0, x^2+y^2\right)\cdot\hat{k}\,d\sigma = \iint_{x^2+y^2\le 1} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1\int_0^{2\pi} r^2 r\,d\theta\,dr = 2\pi\int_0^1 r^3\,dr = 2\pi \left.\frac{r^4}{4}\right]_0^1$$

$$= \boxed{\frac{\pi}{2}}\ \frac{kg}{s}$$ ← "units required!"

Problem #6: $S: x^2+y^2+z^2=4 \quad z \geq 0$

$$C: x^2+y^2=4$$

$$\vec{F} = y\hat{i} - x\hat{j} + \hat{k}$$

- $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = 4\int_0^{2\pi} (\sin t, -\cos t)(-\sin t, \cos t)\,dt = -8\pi$

$$\vec{r} = 2\overrightarrow{(\cos t, \sin t, 0)}$$

$$d\vec{r} = 2\overrightarrow{(-\sin t, \cos t, 0)}$$

- $\iint_S \text{Curl}\,\vec{F}\cdot\vec{n}\,d\sigma = \int_0^{\pi/2}\int_0^{2\pi} (0,0,-2)\cdot\frac{(x,y,z)}{2}\, 2^2 \sin\varphi \, d\theta\, d\varphi$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & 1 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-1-1) = -2\hat{k}$$

$$\vec{n} = \frac{\overrightarrow{(x,y,z)}}{2} \qquad z = 2\cos\varphi \,, \quad d\sigma = 2^2 \sin\varphi\, d\theta\, d\varphi$$

$$= 2\pi\int_0^{\pi/2} -8\cos\varphi\sin\varphi\, d\varphi = -16\pi \left.\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} = -8\pi$$

$u = \sin\varphi$

$du = \cos\varphi\, d\varphi$

Problem #7: $\vec{F} = M\hat{i} + N\hat{j}$

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- $\mathcal{C}$ takes A to B.
- Let $(x(t), y(t))$, $a \le t \le b$ parameterize $\mathcal{C}$

$(x(b), y(b)) = B$, $(x(a), y(a)) = A$.

- $$\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \int_{\mathcal{C}} \overrightarrow{(f_x, f_y)}\,\overrightarrow{(x'(t), y'(t))}\,dt = \int_a^b \nabla f\cdot(x', y')\,dt$$

$$= \int_a^b \frac{d}{dt} f(x(t), y(t))\,dt = f(B) - f(A)$$

ⓑ $$\operatorname{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}\cdot(0) + \hat{k}\left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)$$

$$= 0$$