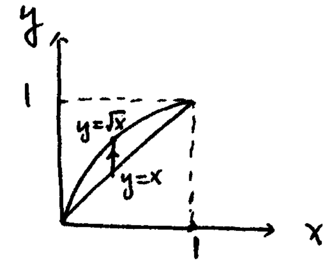
Final Exam

MATH 21 D

Fall '06 — Temple

Solutions



#1 $\int_0^1 \int_x^{\sqrt{x}} x^2 y \, dy \, dx = \int_0^1 \frac{x^2 y^2}{2} \Big|_{y=x}^{y=\sqrt{x}} dx$

$= \int_0^1 \left(\frac{x^2 (\sqrt{x})^2}{2} - \frac{x^2 \cdot x^2}{2} \right) dx$

$= \int_0^1 \left(\frac{x^3}{2} - \frac{x^4}{2} \right) dx = \frac{x^4}{8} - \frac{x^5}{10} \Big|_0^1 = \frac{1}{8} - \frac{1}{10} = \frac{1}{40}$

#2 parametrization of C : $(x,y,z) = (1,-1,0) + t(2-1, 0-(-1), 1-0)$, $\boxed{0 \le t \le 1}$

$= (1+t, -1+t, t)$

$\Rightarrow (dx, dy, dz) = (dt, dt, dt)$

$\Rightarrow \int_C \vec{F} \cdot \vec{T} \, ds = \int_C M\,dx + N\,dy + P\,dz$

$= \int_C y\,dx + 2x\,dy - z\,dz$

$= \int_0^1 \big((-1+t) + 2(1+t) - t \big) dt$

$= \int_0^1 (2t+1)\,dt$

$= t^2 + t \Big|_0^1 = 1^2 + 1 = 2$

#3 $M = \iint_S \delta \, d\sigma = \int_{-1}^1 \int_{-1}^1 \sqrt{1+4x^2+4y^2} \, \alpha \, dx\,dy$, where $\alpha = \sqrt{f_x^2 + f_y^2 + 1}$

$f(x,y) = x^2 + y^2 \Rightarrow \alpha = \sqrt{(2x)^2 + (2y)^2 + 1} = \sqrt{4x^2+4y^2+1}$

$\therefore M = \int_{-1}^1 \int_{-1}^1 \sqrt{1+4x^2+4y^2} \sqrt{4x^2+4y^2+1} \, dx\,dy$

$= \int_{-1}^1 \int_{-1}^1 (1+4x^2+4y^2) \, dx\,dy$

$= \int_{-1}^1 \left[x + \frac{4x^3}{3} + 4y^2 x \right]_{x=-1}^{x=1} dy = \int_{-1}^1 \left(1 + \frac{4}{3} + 4y^2 - \left(-1 - \frac{4}{3} - 4y^2 \right) \right) dy$

$= \int_{-1}^1 \left(2 + \frac{8}{3} + 8y^2 \right) dy = \int_{-1}^1 \left(\frac{14}{3} + 8y^2 \right) dy = \frac{14}{3} y + \frac{8y^3}{3} \Big|_{-1}^1$

$= 2\left(\frac{14}{3} + \frac{8}{3} \right) = \frac{44}{3}$

#4 $\int_C M\,dx + N\,dy = \iint_R N_x - M_y \; dx\,dy$

- Choose $M = -y$, $N = 0$: By Green's Thm, $\int_C -y\,dx = \iint_R 1 \cdot dx\,dy = A$
- Choose $N = x$, $M = 0$: By Green's Thm, $\int_C x\,dy = \iint_R 1 \cdot dx\,dy = A$

#5 $\vec{r} = (u+v)\,\underset{\sim}{i} + (v^2 - u^2)\,\underset{\sim}{j} + 3\,\underset{\sim}{k}$, $0 \le u \le 1$, $0 \le v \le 1$

$\vec{r}_u = \underset{\sim}{i} - 2u\,\underset{\sim}{j}$, $\vec{r}_v = \underset{\sim}{i} + 2v\,\underset{\sim}{j}$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2u & 0 \\ 1 & 2v & 0 \end{vmatrix} = (2v+2u)\,\underset{\sim}{k} \quad \Rightarrow \quad \vec{n} = \underset{\sim}{k}$$

$|\vec{r}_u \times \vec{r}_v| = 2(u+v)$, $\vec{F} = x\,\underset{\sim}{k} = (u+v)\,\underset{\sim}{k}$

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_0^1 \int_0^1 \vec{F} \cdot \underset{\sim}{k}\,(2u+2v)\,du\,dv$$

$$= \int_0^1 \int_0^1 2(u+v)^2\,du\,dv$$

$$= 2\int_0^1 \int_0^1 u^2 + 2uv + v^2\,du\,dv$$

$$= 2\int_0^1 \left. \frac{u^3}{3} + u^2 v + uv^2 \right]_{u=0}^{u=1} dv$$

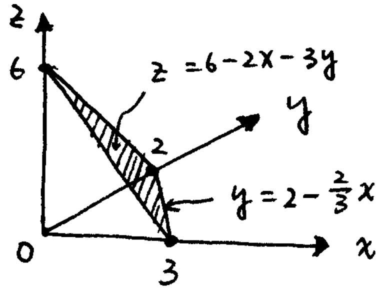
$$= 2\int_0^1 \left(\frac{1}{3} + v + v^2\right) dv$$

$$= 2\left(\frac{v}{3} + \frac{v^2}{2} + \frac{v^3}{3}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{3}\right) = \frac{7}{3}$$

#6 $\displaystyle\int_0^2 \int_0^{6-3y} \int_0^{3-\frac{1}{2}z-\frac{3}{2}y} dx\,dz\,dy$

sketch: $x = 3 - \frac{1}{2}z - \frac{3}{2}y \iff z = 6 - 2x - 3y$



re-iterate: $\displaystyle\int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{6-2x-3y} dz\,dy\,dx$

#7 (a) $\int_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds$ is the length of C_ε weighted with values of the component of $\vec{F}$ in the direction of $\vec{T}$ $\approx$ " the amount of $\vec{F}$ pointing around C_ε " (Anything in the right ballpark is OK)

(b) When $\varepsilon \ll 1$, $\text{Curl}\,\vec{F}\cdot\vec{n} \approx \text{Curl}\,\vec{F}(\underset{\sim}{x_0})\cdot\vec{n}$.

So $\displaystyle\int_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds \approx \text{Curl}\,\vec{F}(\underset{\sim}{x_0})\cdot\vec{n} \iint_{D_\varepsilon} d\sigma$

$= \text{Curl}\,\vec{F}(\underset{\sim}{x_0})\cdot\vec{n}\ (\text{Area of } D_\varepsilon)$

$\therefore \text{Curl}\,\vec{F}(\underset{\sim}{x_0})\cdot\vec{n} \approx \dfrac{1}{(\text{Area of } D_\varepsilon)} \displaystyle\int_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds$

Taking $\varepsilon \to 0$, $\text{Curl}\,\vec{F}(\underset{\sim}{x_0})\cdot\vec{n} = \displaystyle\lim_{\varepsilon\to 0} \frac{1}{(\text{Area of } D_\varepsilon)} \int_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds$

= " circulation per area around axis $\vec{n}$ "

(c) $\text{Curl}\,\vec{F}\cdot\vec{n} = |\text{Curl}\,\vec{F}|\,|\vec{n}|\cos\theta$ takes its maximum value when $\theta = 0$.

$\Rightarrow \vec{n} = \dfrac{\text{Curl}\,\vec{F}}{|\text{Curl}\,\vec{F}|}$

$\Rightarrow$ Maximal circulation per area occurs around axis $\text{Curl}\,\vec{F}$.

(d) $|\text{Curl}\,\vec{F}| = \text{Curl}\,\vec{F}\cdot\dfrac{\text{Curl}\,\vec{F}}{|\text{Curl}\,\vec{F}|}$ = maximal circulation per area

(#8) $\vec{F} = y\,\underset{\sim}{i} + (x+2z)\,\underset{\sim}{j} + (2y-1)\,\underset{\sim}{k}$

ⓐ $\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ y & x+2z & 2y-1 \end{vmatrix} = \underset{\sim}{i}(2-2) - \underset{\sim}{j}(0-0) + \underset{\sim}{k}(1-1)$

$= 0$ ✓

ⓑ $f(x,y,z) = \int_x y\,dx = yx + g(y,z)$

$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y,z) = x + 2z \Rightarrow \frac{\partial g}{\partial y} = 2z$

$\Rightarrow g(y,z) = 2zy + h(z)$

$\frac{\partial f}{\partial z} = 0 + \frac{\partial g}{\partial z} = 2y + h'(z) = 2y - 1 \Rightarrow h'(z) = -1$

$\Rightarrow h(z) = -z$

$f(x,y,z) = xy + g(y,z)$

$= xy + 2yz + h(z) = \boxed{xy + 2yz - z}$.

ⓒ $\int_P^Q \vec{F}\cdot\vec{T}\,ds = f(Q) - f(P) = f(0,1,0) - f(1,0,2)$

$= 0 - (-2) = \boxed{2}$

#9 $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \operatorname{div}\vec{F}\,dv$

(a) $$\underbrace{\frac{d}{dt}\underbrace{\iiint_D \delta(x,y,z,t)\,dx\,dy\,dz}_{\text{total mass in D at time t}}}_{\text{rate at which mass changes in D}} = \underbrace{-\underbrace{\iint_S \vec{F}\cdot\vec{n}\,d\sigma}_{\text{rate at which mass leaves through boundary S}}}_{\text{rate at which mass enters D}}$$

- (a) expresses that the rate of change of mass in D is due entirely to the flux of mass thru boundary S
  - $\equiv$ no mass created or destroyed
  - $\equiv$ conservation of mass

(b) From (a) we get : (using div. thm)

$$\iiint_D \delta_t(x,y,z,t)\,dv = -\iiint_D \operatorname{div}\vec{F}\,dv$$

Putting these on the same side and using additive property of integrals:

$$\iiint_D \delta_t + \operatorname{div}\vec{F}\,dv = 0 \qquad (*)$$

By conservation of mass, $(*)$ holds in every volume D.

We must have $\delta_t + \operatorname{div}\vec{F} = 0$.

$$\Rightarrow \boxed{\delta_t + \operatorname{div}\delta\vec{v} = 0}$$