Name: Solutions

Student ID#: ____________

Section: ____________

Final Exam

Friday March 21

MAT 21D, Temple, Winter 2014

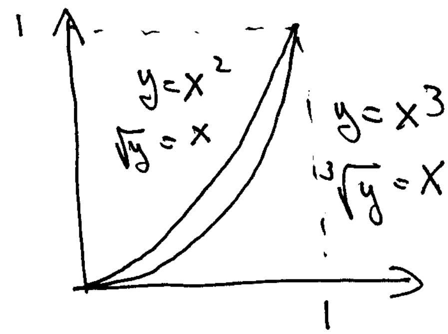
Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (25pts): **(a)** Sketch the region of integration $\mathbf{R}_{xy}$ and evaluate the iterated integral

$$\int_0^1 \int_{x^3}^{x^2} xy^2 \, dy dx. \tag{1}$$

$$= \int_0^1 \left. \frac{x y^3}{3} \right]_{y=x^3}^{y=x^2} dx = \int_0^1 \frac{x x^6 - x x^9}{3} \, dx$$



$$= \frac{1}{3}\int_0^1 x^7 - x^{10} \, dx = \frac{1}{3}\left(\frac{x^8}{8} - \frac{x^{11}}{11} \right)\Big|_0^1$$

$$= \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11} \right)$$

(b) Rewrite (1) with order of integration reversed. (Do not re-evaluate).

$$\int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} xy^2 \, dx \, dy$$

(c) Let $\mathcal{R}_{xy}$ denote the region $x^2 + y^2 \leq 1$. Use polar coordinates to evaluate the integral:

$$\int\int_{\mathcal{R}_{xy}} (x^2 + y^2) dx dy.$$

$$= \int_0^{2\pi} \int_0^1 r^2 r \, dr d\theta = \int_0^{2\pi} \int_0^1 r^3 \, dr d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4} \Big]_0^1 d\theta = \frac{1}{4} \int_0^{2\pi} d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

Problem #2 (25pts): A particle of mass $m = 2\ kg$ moves along a trajectory given by

$$\mathbf{r}(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j} + 2t\ \mathbf{k},$$

where t is in *seconds* and $\mathbf{r}$ is in *meters*. At each time t find the following, give the correct dimensions, or say its dimensionless:

(a) The velocity vector $\mathbf{v}(t)$ $= \vec{r}\,'(t) = -\cos t\ \underset{\sim}{i} - \sin t\ \underset{\sim}{j} + 2\underset{\sim}{k}$ $\quad \frac{L}{T}$

(b) The speed $v(t)$ $= \|\vec{v}\| = \sqrt{\cos^2 t + \sin^2 t + 4} = \sqrt{5}$ $\quad \frac{L}{T}$

(c) The acceleration vector $\mathbf{a}(t)$ $= \vec{v}\,'(t) = \sin t\ \underset{\sim}{i} - \cos t\ \underset{\sim}{j}$ $\quad \frac{L}{T^2}$

(d) The unit tangent vector $\mathbf{T}(t)$ $= \frac{\vec{v}}{\|\vec{v}\|} = \frac{-\cos t}{\sqrt{5}}\underset{\sim}{i} - \frac{\sin t}{\sqrt{5}}\underset{\sim}{j} + \frac{2}{\sqrt{5}}\underset{\sim}{k}$

1

(e) The unit normal $\mathbf{N}(t)$ $= \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \left(\frac{\sin t}{\sqrt{5}}\underset{\sim}{i} - \frac{\cos t}{\sqrt{5}}\underset{\sim}{j}\right)\frac{1}{\sqrt{\frac{\sin^2 t + \cos^2 t}{5}}}$

$= \sin t\ \underset{\sim}{i} - \cos t\ \underset{\sim}{j}$ $\quad 1$

(f) The curvature $\kappa(t)$. $= \dfrac{1}{\|\vec{v}\|}\left\|\dfrac{d\vec{T}}{dt}\right\| = \dfrac{1}{\sqrt{5}}\cdot\dfrac{1}{\sqrt{5}} = \dfrac{1}{5}$ 1

(g) Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$, the lengths of the components $\mathbf{a}(t)$ in directions of $\mathbf{T}$ and $\mathbf{N}$.

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa \vec{N} = a_T \vec{T} + a_N \vec{N}$$

$$a_T = \vec{a}\cdot\vec{T} = \overrightarrow{(\sin t, -\cos t, 0)} \cdot \overrightarrow{\left(-\frac{\cos t}{\sqrt{5}}, -\frac{\sin t}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)} = 0$$

$$a_N = \vec{a}\cdot\vec{N} = \overrightarrow{(\sin t, -\cos t, 0)} \cdot \overrightarrow{(\sin t, -\cos t, 0)} = 1$$

$[a_T] = [a_N] = [\vec{a}]$

$= \dfrac{L}{T^2}$

(h) Find the arclength from $t = 0$ to $t = 1$.

$$\int_0^1 \frac{ds}{dt}\,dt = \int_0^1 \sqrt{5}\,dt = \sqrt{5}$$ L

Problem #3 (25pts): **(a)** Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field, where M, N, P are assumed to be given functions of (x, y, z). Use Leibniz's substitution principle to show the following are equal: (Here $\mathbf{r}(t)$ denotes any parameterization of curve C.)

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{v}\,dt = \int_C Mdx + Ndy + Pdz.$$

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \|\vec{v}\| = \frac{ds}{dt} \qquad \vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\Rightarrow ds = \|\vec{v}\|dt \Rightarrow \vec{T}\,ds = \frac{\vec{v}}{\|\vec{v}\|}\|\vec{v}\|dt = \vec{v}\,dt \qquad \boxed{\vec{T}ds = \vec{v}dt}$$

$$\boxed{d\vec{r} = \vec{v}\,dt} \qquad \vec{F}\cdot\vec{v}dt = (M, N, P)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)dt = Mdx + Ndy + Pdz$$

$$\therefore \int_C \vec{F}\cdot\underbrace{\vec{T}ds}_{\vec{v}dt} = \int_C \vec{F}\cdot\underbrace{\vec{v}\,dt}_{d\vec{r}} = \int_C \vec{F}\cdot d\vec{r} = \int_C \underbrace{Mdx + Ndy + Pdz}_{\vec{F}\cdot\vec{v}\,dt}$$

(b) Give the formulas for $Curl\mathbf{F}$ and $Div\mathbf{F}$.

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \underset{\sim}{i}(P_y - N_z) - \underset{\sim}{j}(P_x - M_z) + \underset{\sim}{k}(N_x - M_y)$$

$$\text{Div}\,\vec{F} = M_x + N_y + P_z$$

Problem #4 (25pts): Show that if $\mathbf{F} = \frac{1}{2}y^2\,\mathbf{i} + x(y+1)\,\mathbf{j}$, then for any closed curve $\mathcal{C}$ in the (x, y)-plane, $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$=Area enclosed by $\mathcal{C}$.

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = \iint_{A} N_x - M_y \, dx\,dy$$

Green's Thm

$$= \iint_{A} (y+1) - (y) \, dx\,dy$$

$$= \iint_{A} dx\,dy = \text{area of } A.$$

Problem #5 (25pts): Let $\mathbf{F} = -\frac{y}{r^2}\,\mathbf{i} + \frac{x}{r^2}\,\mathbf{j}$, where $r = \sqrt{x^2+y^2}$.
(a) Find $Curl(\mathbf{F})$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{\hat{i}} & \underset{\sim}{\hat{j}} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_t \\ -\frac{y}{r^2} & +\frac{x}{r^2} & 0 \end{vmatrix} = \underset{\sim}{i}\cdot 0 + \underset{\sim}{j}\cdot 0 + \underset{\sim}{k}\left(\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right)\right)$$

$$\left.\begin{aligned} \frac{\partial}{\partial x}\frac{x}{r^2} &= \frac{1}{r^2} - \frac{2x^2}{r^3}\cdot\frac{1}{r} \\ \frac{\partial}{\partial y}\frac{y}{r^2} &= \frac{1}{r^2} - \frac{2y^2}{r^3}\,\frac{1}{r} \end{aligned}\right\} \quad \frac{\partial}{\partial x}\frac{x}{r^2} + \frac{\partial}{\partial y}\frac{y}{r^2} = \frac{2}{r^2} - 2\,\frac{x^2+y^2}{r^4}$$

$$= \frac{2}{r^2} - \frac{2}{r^2} = 0$$

(b) Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ by explicit parameterization, where C is the unit circle oriented counterclockwise.

$x = \cos t$

$y = \sin t$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \vec{F}\cdot\vec{v}\,dt$$

$\vec{r}(t) = \overrightarrow{(\cos t, \sin t)}$

$\vec{r}\,'(t) = \overrightarrow{(-\sin t, \cos t)} = \vec{v}$

$$= \int_0^{2\pi} \overrightarrow{(-\sin t, \cos t)} \cdot \overrightarrow{(-\sin t, \cos t)}\,dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt$$

$$= \int_0^{2\pi} dt = 2\pi$$

Problem #6 (25pts): Let $\mathcal{R}_{xy}$ denote the area bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Use the substitution $x = au,\ y = bv$ to evaluate the integral

$$\int\int_{\mathcal{R}_{xy}} \frac{x^2}{a^2}+\frac{y^2}{b^2}\,dxdy.$$

Set $\vec{r}(u,v) = au\,\underset{\sim}{i} + bv\,\underset{\sim}{j}$ $\quad R_{uv}: u^2+v^2 \le 1$

$\vec{r}_u = \overrightarrow{(a,0)} \quad \vec{r}_v = \overrightarrow{(0,b)}$

$$\|\vec{r}_u \times \vec{r}_v\| = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a & 0 & 0 \\ 0 & b & 0 \end{vmatrix} = \|\underset{\sim}{k}\cdot ab\| = ab$$

$$\iint_{R_{xy}} \frac{x^2}{a^2}+\frac{y^2}{b^2}\,dxdy = ab\iint_{R_{uv}} u^2+v^2\,du\,dv = ab\int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta$$

$$= \frac{ab}{4}\cdot 2\pi = \boxed{\frac{ab\pi}{2}}$$

Problem #7 (25pts): (a) State Stokes Theorem for a disk $D_\epsilon(\mathbf{x_0})$ of radius ϵ and center $\mathbf{x}_0 = (x_0, y_0, z_0)$ and normal $\mathbf{n}$, and use this to derive the meaning of $Curl\mathbf{F}\cdot\mathbf{n}$ at $\mathbf{x}_0$ as *the circulation in* $\mathbf{F}$ *per area around axis* $\mathbf{n}$ *at* $\mathbf{x_0}$.

$$\oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds = \iint_{D_\varepsilon} Curl\vec{F}\cdot\vec{n}\,d\sigma \underset{\varepsilon<<1}{\approx} Curl\vec{F}\cdot\vec{n}(\underset{\sim}{x}_0)\,Area(D_\varepsilon)$$

$$\therefore\ Curl\vec{F}\cdot\vec{n}\Big|_{\underset{\sim}{x}_0} = \lim_{\varepsilon\to 0}\frac{1}{Area\,D_\varepsilon}\int_{C_\varepsilon}\vec{F}\cdot\vec{T}\,ds$$

$= $ "Circulation per area around $\vec{n}$"

(b) Let $\mathbf{F} = y\mathbf{i} - x^2\mathbf{j} + xy\mathbf{k}$. Find the circulation per area around the axis $\mathbf{j}-\mathbf{k}$ at the point $P_0 = (-1, 1, 2)$.

$$Curl\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x^2 & xy \end{vmatrix} = \underset{\sim}{i}(x-0) - \underset{\sim}{j}(y-0) + \underset{\sim}{k}(\underset{-2x-1}{\cancel{2x}})$$

$$= \overrightarrow{(x, -y, \underset{-2x-1}{\cancel{2x}})}$$

$$Curl\vec{F}\cdot\vec{n} = \overrightarrow{(\underset{x}{\cancel{x}}, -y, \underset{-2x-1}{\cancel{2x}})}\cdot\overrightarrow{(0,1,-1)}\frac{1}{\sqrt{2}} = \frac{-y+2x+1}{\sqrt{2}}$$

$$Curl\vec{F}\cdot\vec{n}\Big|_{(-1,1,2)} = \frac{\overset{-1-2+1}{\cancel{-1+2}}}{\sqrt{2}} = \cancel{\frac{1}{\sqrt{2}}}\ -\sqrt{2}$$

(c): Use (**a**) to show that the maximal circulation per area at $\mathbf{x}_0$ occurs around an axis parallel to the $Curl\mathbf{F}$, and that the length of $Curl\mathbf{F}$ is that maximal circulation per area.

$$\mathrm{Curl}\vec{F}\cdot\vec{n} = \|\mathrm{Curl}\vec{F}\|\cdot\|\vec{n}\|\cos\theta = \|\mathrm{Curl}\vec{F}\|\cos\theta$$

where $\theta = \angle$ betw $\mathrm{Curl}\vec{F}$ & $\vec{n}$

Max when $\cos\theta = 1 \Rightarrow \theta = 0$

$$\Rightarrow \vec{n} = \frac{\mathrm{Curl}\vec{F}}{\|\mathrm{Curl}\vec{F}\|}$$

$$\text{So Max } \mathrm{Curl}\vec{F}\cdot\vec{n} = \mathrm{Curl}\vec{F}\cdot\frac{\mathrm{Curl}\vec{F}}{\|\mathrm{Curl}\vec{F}\|} = \|\mathrm{Curl}\vec{F}\|$$

Conclude: $\mathrm{Curl}\vec{F}\cdot\vec{n}$ max'al when $\vec{n}$ in direction of $\mathrm{Curl}\vec{F}$ & max'al value is $\|\mathrm{Curl}\vec{F}\|$

Problem #8 (25pts): Recall the Divergence Theorem

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_{\mathcal{V}} Div\mathbf{F}\,dV, \qquad (2)$$

where $\mathcal{V}$ is the sphere of radius $\rho = 1$ and $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Verify (2) by explicitly evaluating both sides using spherical coordinates

$$x = \rho\cos(\theta)\sin(\phi); \quad y = \rho\sin(\theta)\sin(\phi); \quad z = \rho\cos(\phi)$$

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_S d\sigma = \iint_{R_{\phi\theta}} \|\vec{r}_\varphi \times \vec{r}_\theta\|\,d\varphi\,d\theta = *$$

$\vec{F}\cdot\vec{n} = 1$ on S

$$\vec{r}(\varphi,\theta) = \cos\theta\sin\varphi\,\underset{\sim}{i} + \sin\theta\sin\varphi\,\underset{\sim}{j} + \cos\varphi\,\underset{\sim}{k}$$

$$\vec{r}_\varphi = \cos\theta\cos\varphi\,\underset{\sim}{i} + \sin\theta\,\cancel{\sin\varphi}\,\overset{}{\underset{\cos\varphi}{}}\,\underset{\sim}{j} - \sin\varphi\,\underset{\sim}{k}$$

$$\vec{r}_\theta = -\sin\theta\sin\varphi\,\underset{\sim}{i} + \cos\theta\sin\varphi\,\underset{\sim}{j}$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \cos\theta\cos\varphi & \sin\theta\,\cancel{\sin\varphi}\,\cos\varphi & -\sin\varphi \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \end{vmatrix} = \underset{\sim}{i}(\cos\theta\sin^2\varphi) + \underset{\sim}{j}(\sin\theta\sin^2\varphi) + k(\cos\varphi\sin\varphi)$$

$$\|\vec{r}_\varphi \times \vec{r}_\theta\|^2 = \cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \cos^2\varphi\sin^2\varphi$$

$$= \sin^4\varphi + \cos^2\varphi\sin^2\varphi = \sin^2\varphi(\sin^2\varphi + \cos^2\varphi) = \sin^2\varphi$$

$$* = \int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta = 2\pi(-\cos\varphi)\Big|_0^{\pi} = 4\pi$$

$$\iiint_V \mathrm{Div}\,\vec{F}\,dV = 3\iiint_V dV$$

$$= 3\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= 3\int_0^{\pi}\int_0^{2\pi} \left.\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=1} \sin\varphi \, d\theta\, d\varphi$$

$$= 2\pi\int_0^{\pi} \sin\varphi\, d\varphi = 2\pi\left[-\cos\varphi\right]_0^{\pi} = \boxed{4\pi} \checkmark$$