Name: Solutions

Student ID#: __________

Section: __________

Final Exam

Wednesday March 20

MAT 21D, Temple, Winter 2013

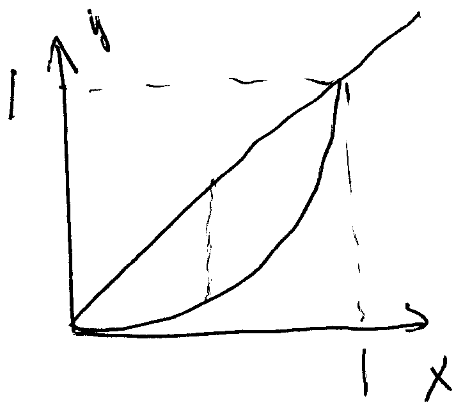
Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (25pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ determined by the iterated integral

$$\int_0^1 \int_{x^2}^{x} xy^2 \, dydx. \tag{1}$$

and evaluate it.



$$\int_0^1 \int_{x^2}^{x} xy^2\,dydx = \int_0^1 x \left.\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \frac{x^4}{3} - \frac{x^7}{3}\,dx = \frac{1}{3}\left(\frac{x^5}{5} - \frac{x^8}{8}\right)\Big]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right) = \frac{1}{3}\,\frac{8-5}{40} = \frac{1}{40}$$

(b) Rewrite (1) with order of integration reversed, and evaluate it.

$$\int_0^1 \int_y^{\sqrt{y}} xy^2\,dxdy = \int_0^1 y^2 \left.\frac{x^2}{2}\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 y^2\left(\frac{(\sqrt{y})^2}{2} - \frac{y^2}{2}\right) dy = \int_0^1 y^2\left(\frac{y}{2} - \frac{y^2}{2}\right) dy$$

$$= \int_0^1 \frac{y^3}{2} - \frac{y^4}{2}\,dy = \frac{1}{2}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\Big]_0^1 = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{40}$$

Problem #2 (25pts): A particle moves along a trajectory given by

$$\mathbf{r}(t) = 2\sin t\ \mathbf{i} + 2\cos t\ \mathbf{j} - t\ \mathbf{k}.$$

Find the following:

(a) The velocity vector $\mathbf{v}(t)$

$\vec{v}(t) = \vec{r}\,'(t) = 2\cos t\ \underset{\sim}{i} - 2\sin t\ \underset{\sim}{j} - k$

(b) The speed $v(t)$

$v(t) = \|\vec{v}(t)\| = \sqrt{4\cos^2 t + 4\sin^2 t + 1} = \sqrt{5}$

(c) The unit tangent vector $\mathbf{T}(t)$

$\vec{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|} = \dfrac{1}{\sqrt{5}}\left(2\cos t\ \underset{\sim}{i} - 2\sin t\ \underset{\sim}{j} - \underset{\sim}{k}\right)$

(d) The unit normal $\mathbf{N}(t)$

$\vec{N}(t) = \dfrac{\frac{d\vec{T}}{dt}}{\left\|\frac{d\vec{T}}{dt}\right\|}$ $\qquad \dfrac{d\vec{T}}{dt} = -\dfrac{2}{\sqrt{5}}\sin t\ \underset{\sim}{i} - \dfrac{2}{\sqrt{5}}\cos t\ \underset{\sim}{j}$

$\left\|\dfrac{d\vec{T}}{dt}\right\| = \dfrac{2}{\sqrt{5}}$

$\vec{N}(t) = -\sin t\ \underset{\sim}{i} - \cos t\ \underset{\sim}{j}$

(e) Find the acceleration vector $\mathbf{a}(t)$

$\vec{a} = \vec{r}''(t) = \vec{v}'(t) = -2\sin t \, \underset{\sim}{i} - 2\cos t \, \underset{\sim}{j}$

(f) Find a_T and a_N such that $\mathbf{a}(t) = a_T\mathbf{T} + a_N\mathbf{N}$.

$a_T = \vec{a} \circ \vec{T} = 0$, $a_N = \vec{a} \cdot \vec{N} = 2$

(g) Find the arclength from $t = 0$ to $t = 1$. $ds = \|\vec{v}(t)\| \, dt$

$\int_0^1 \|\vec{v}\| \, dt = \int_0^1 \sqrt{5} \, dt = \sqrt{5}(1-0) = \sqrt{5}$

Problem #3 (25pts): Let

$$r(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

(a) Derive the formula

$$\frac{\partial}{\partial x} r(x, y, z) = \frac{x}{r}.$$

$$\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \checkmark$$

(b) Write the corresponding formula for y and z.

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(c) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $\mathbf{F} = -\frac{\mathbf{r}}{r^3}$ be the gravitational force field. Use (a) to show that $\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3}$, (i.e., $\mathbf{F}$ is conservative.)

$$\frac{\partial}{\partial x} \frac{1}{r} = -\frac{1}{r^2} \cdot \frac{x}{r} = -\frac{x}{r^3}$$

$$\frac{\partial}{\partial y} \frac{1}{r} = -\frac{1}{r^2} \cdot \frac{y}{r} = -\frac{y}{r^3}$$

$$\frac{\partial}{\partial z} \frac{1}{r} = -\frac{1}{r^2} \frac{z}{r} = -\frac{z}{r^3}$$

$$\nabla \frac{1}{r} = -\frac{x}{r^3} \underset{\sim}{i} - \frac{y}{r^3} \underset{\sim}{j} - \frac{z}{r^3} \underset{\sim}{k} = -\frac{1}{r^3} \vec{r}$$

(d) Let $\mathcal{C}$ be any curve taking $A = (1, -1, 2)$ to $B = (-3, 0, 4)$. Evaluate the line integral $\int_{\mathcal{C}} F \cdot \mathbf{T} ds$, (the work done by F along $\mathcal{C}$).

$$\int_{\mathcal{C}} F \cdot T \, ds = \left. \frac{1}{r} \right]_A^B = \frac{-1}{\sqrt{1^2+1^2+2^2}} + \frac{1}{\sqrt{3^2+0^2+4^2}}$$

$$= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{5^2}} = \cancel{\frac{1}{2} - \frac{1}{5}} = \cancel{\frac{3}{10}}$$

Problem #4 (25pts): (a) Use the Leibniz substitution principle to show that in general, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a parameterization of curve C for $a \leq t \leq b$, and $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field, then the following are equal:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_C M dx + N dy + P dz.$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F} \cdot \frac{\vec{v}}{\cancel{\|\vec{v}\|}} \cancel{\|\vec{v}\|} dt = \int_C \vec{F} \cdot \vec{v}\, dt$$

$\vec{r}(t)$

$a \leq t \leq b$

$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$

$ds = \|\vec{v}\| dt$

$$d\vec{r} = \frac{d\vec{r}}{dt} dt = \vec{v}\, dt$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$\vec{v}\, dt = \left(\frac{dx}{dt} \underset{\sim}{i} + \frac{dy}{dt} \underset{\sim}{j} + \frac{dz}{dt} \underset{\sim}{k} \right) \cdot dt$$

$$= \overrightarrow{(dx, dy, dz)}$$

$$\int_C \vec{F} \cdot \vec{v}\, dt = \int_C \vec{F} \cdot \overrightarrow{(dx, dy, dz)} = \int_C M dx + N dy + P dz$$

#4 **(b)** Assume further that $\mathbf{r}(t)$, $a \le t \le b$ is a parameterization of C which describes the motion of an object subject to the total force $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a} = \mathbf{r}''(\mathbf{t})$. Prove that

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \frac{1}{2}mv(\mathbf{r}(b))^2 - \frac{1}{2}mv(\mathbf{r}(a))^2.$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F} \cdot \vec{v}\, dt = \int_a^b m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt$$

$$= \int_a^b m \frac{1}{2} \frac{d}{dt}(\vec{v} \cdot \vec{v})\, dt$$

$$= \frac{1}{2} m\, \vec{v} \cdot \vec{v} \Big]_{t=a}^{t=b}$$

$$= \frac{1}{2} m v^2(b) - \frac{1}{2} m v^2(a)$$

Problem #5 (25pts): Let $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z)\mathbf{k}$. Find:

(a) Curl $\mathbf{F}$ $= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy & 0 & x^2+z \end{vmatrix} = i(0-0) - j(2x-0) + k\overset{-2x}{(0)}$

$= -2x\, j$ -2xk

(c) Find the circulation per area in **F** around an axis aligned with the direction $\mathbf{v} = \mathbf{i} + \mathbf{j}$. at the point $(1,-2,-1)$.

$\vec{n} = \frac{1}{\sqrt{2}}(1,1,0)$

$\text{Curl}\vec{F} \cdot \vec{n} = (0, -2x, -2x) \cdot \frac{1}{\sqrt{2}}(1,1,0) = -\frac{2}{\sqrt{2}}x$

$\cancel{x = -2 \implies = -\frac{2}{\sqrt{2}} \cdot (-2) = \boxed{\frac{4}{\sqrt{2}}}}$

x=1 so answer is -2/sqrt{2}

Problem #6 (25pts): Let $\mathcal{S}$ be the surface in $\mathcal{R}^3$ which is the image of $0 \le u \le 1$, $0 \le v \le 2$ under the parameterization $\mathbf{r}(u,v) = u^2\mathbf{i} + v^2\mathbf{j} - 2uv\mathbf{k}$. Evaluate the flux integral

$$\int\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where $\mathbf{F}$ is the vector field $\mathbf{F} = z\,\mathbf{k}$ and $\mathbf{n}$ points toward negative x.

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_{R_{uv}} \vec{F} \cdot \underbrace{\frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}}}_{\text{check } \pm \vec{n}} \cancel{\|\vec{r}_u \times \vec{r}_v\|}\, du\, dv$$

$$\vec{r}_u = 2u\,\underset{\sim}{i} - 2v\,\underset{\sim}{k} \qquad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2u & 0 & -2v \\ 0 & 2v & -2u \end{vmatrix}$$

$$\vec{r}_v = 2v\,\underset{\sim}{j} - 2u\,\underset{\sim}{k}$$

$$\vec{n} = -\frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \qquad = \underset{\sim}{i}(0 + 4v^2) + \underset{\sim}{j}(4u^2) + \underset{\sim}{k}(4uv)$$

points toward positive x

$$-\vec{F} \cdot \vec{r}_u \times \vec{r}_v = -\overrightarrow{(0,0,z)} \cdot \overrightarrow{(4v^2, +4u^2, 4uv)} = -4uvz$$

$z = -2uv$

$$= +8u^2v^2$$

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_0^2 \int_0^1 +8u^2v^2\, du\, dv = \int_0^2 +8v^2 \left.\frac{u^3}{3}\right]_0^1 dv$$

$$= +\frac{8}{3} \left.\frac{v^3}{3}\right]_0^2 = +\frac{8}{3}\,\frac{8}{3} = \boxed{+\frac{64}{9}}$$

Problem #7 (25pts): Verify Stokes Theorem:

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int\int_S Curl\mathbf{F}\cdot\mathbf{n}\,d\sigma,$$

where $\mathcal{S}$ is the hemisphere $x^2+y^2+z^2=4,\quad z\geq 0,\quad \mathbf{F}=-y\mathbf{i}+x\mathbf{j}$, and $\mathcal{C}$ is the closed curve $x^2+y^2=4$.

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi}\vec{F}\;\vec{v}\,dt = \int_0^{2\pi} 2\overrightarrow{(-\sin t,\cos t)}\cdot 2\overrightarrow{(-\sin t,\cos t)}\,dt$$

$$\vec{r}(t) = 2\overrightarrow{(\cos t,\sin t)} \qquad = \int_0^{2\pi} 4(\cos^2+\sin^2)\,dt = 8\pi$$

$$\vec{v}(t) = 2\overrightarrow{(-\sin t,\cos t)}$$

$$\iint_S Curl\vec{F}\;\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi/2} Curl\vec{F}\cdot\vec{n}\,\underbrace{2^2\sin\varphi}_{\text{spherical coords}}\,d\varphi\,d\theta$$

$$Curl\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \underset{\sim}{i}(0) - \underset{\sim}{j}(0) + \underset{\sim}{k}\,2 = 2\underset{\sim}{k}$$

$$\vec{n} = \frac{1}{2}\overrightarrow{(x,y,z)} \Rightarrow Curl\vec{F}\cdot\vec{n} = z = 2\cos\varphi$$

$$\iint_S Curl\vec{F}\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi/2} 8\cos\varphi\sin\varphi\,d\varphi\,d\theta = 8\cdot\frac{1}{2}\int_0^{2\pi} d\theta$$

$u = \sin\varphi$

$du = \cos\varphi\,d\varphi \qquad = 8\pi$

Problem #8 (25pts): Consider a fluid flow with $\delta(x,y,z)$the density in $\frac{kg}{m^3}$, $\mathbf{v}(x,y,z)$ the velocity in $\frac{m}{s}$, and let $\mathbf{F} = \delta\mathbf{v}$ be the mass flux vector, given by $\mathbf{F}(x,y,z) = x^2yz\mathbf{i} + (x^2+y^2)\mathbf{k}$.

(a) Find $\mathbf{Div\,F}$. $= 2xyz + 0 = 2xyz$

(c) Find the rate at which mass is passing upward through the disk $x^2+y^2 \leq 1$, $z = 0$.

$$\frac{d\,\text{mass}}{d\,\text{time}} = \iint_{S=\text{Disk}} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S} \vec{F}\cdot\underset{\sim}{k}\,d\sigma$$

$$= \iint_{S} x^2+y^2\,d\sigma = \int_0^{2\pi}\int_0^1 r^2 r\,dr\,d\theta$$

polar coords

$x = r\cos\theta$

$y = r\sin\theta$

$d\sigma = dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi} \frac{r^4}{4}\Big]_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} d\theta = \frac{1}{4}2\pi = \boxed{\frac{\pi}{2}}$$