Write solutions on the paper provided. Put your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

Problem 1. (15pts) Evaluate \( \int \int_R x^2y \, dA \) where \( R \) is the region in the \( xy \)-plane bounded between the curves \( y = x \) and \( y = x^{3/2} \).

Solution: Since \( x = x^{3/2} \) implies \( x = 0, 1 \), it follows that the graph of \( y = x \) intersects the graph of \( y = x^{3/2} \) at \((0, 0)\) and \((1, 1)\), and so the region between the graphs consists of the region \( x^{3/2} \leq y \leq x, x \leq 0 \leq 1 \). Thus

\[
\int \int_R x^2y \, dA = \int_0^1 \int_{x^{3/2}}^x x^2y \, dy \, dx = \int_0^1 x^2 \left[ \frac{y^2}{2} \right]_x^{x^{3/2}} \, dx
\]

\[
= \int_0^1 x^2 \left[ \frac{x^2}{2} - \frac{x^3}{2} \right] \, dx = \left[ \frac{x^5}{10} - \frac{x^6}{12} \right]_0^1 = 1/60.
\]
Problem 2. (15pts) Use polar coordinates to evaluate
\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \, dx \, dy. \]

Solution: Using polar coordinates \( r^2 = x^2 + y^2 \), \( dA = rdrd\theta \), and the fact that the region of integration \(-\infty < x, y < \infty\) in \( xy \)-coordinates goes over to the region \( 0 \leq r < \infty \), \( 0 \leq \theta < 2\pi \) in \( r\theta \)-coordinates, we can write:

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r \, dr \, d\theta.
\]

From this the substitution \( u = r^2 \), \( du = dr \, dr \) leads to

\[
= \int_{0}^{2\pi} \left[ -\frac{1}{2} e^{-u} \right]_{0}^{\infty} = \cdots = \pi.
\]
Problem 3. (17pts) Let $\int \int \int_D dV$ be given as an iterated integral in the order $dzdydx$ as

$$\int_0^2 \int_{-3}^{x+3} \int_{0}^{1-x-3y} dzdydx.$$ 

Sketch the region $D$ in $xyz$-space, and set up the iterated integral in the order $\int \int \int dx dz dy$. [DO NOT SOLVE]

Solution: The region of integration is the tetrahedron bounded above by the plane $z = 1 - \frac{1}{2}x - \frac{1}{3}y$, and on the sides by the planes formed by the positive coordinate axes. From this we can obtain:

$$\int_0^2 \int_{-3}^{x+3} \int_{0}^{1-x-3y} dzdydx = \int_0^3 \int_{0}^{-3} \int_{0}^{2z-3y} dx dz dy.$$
Problem 4. (24pts) A thin plate covers the triangular region $R$ between the $x$-axis, the line $y = x$ and the line $x = 1$. The density of the plate is $\delta(x, y) = x + 2y$.

(a) Find the mass $M$ of the plate.

Solution: $M = \int_0^1 \int_0^x (x + 2y) dy dx = \cdots = \frac{2}{3}$.

(b) Set up an iterated integral for $\bar{x}$ and $\bar{y}$ the $x$- and $y$-coordinates of the center of mass. (DO NOT SOLVE.)

Solution:

\[
\bar{x} = \frac{\int_0^1 \int_0^x (x^2 + 2xy) dy dx}{2/3}
\]

\[
\bar{y} = \frac{\int_0^1 \int_0^x (xy + 2y^2) dy dx}{2/3}
\]

(c) Set up an iterated integral for the moment of inertia $I_x$ of the plate about the $x$-axis. (DO NOT SOLVE.)

Solution:

\[
I_x = \int_0^1 \int_0^x (xy^2 + 2y^3) dy dx
\]
Problem 5. (24pts) Consider the iterated integral
\[ \int \int_{R} dA = \int_{0}^{1} \int_{x^2}^{x} \sin(xy) \, dy \, dx. \]

(a) Sketch the region \( R \) in the \( xy \)-plane.

**Solution:** The region is \( x^2 \leq y \leq x, \quad 0 \leq x \leq 1. \)

(b) Let \( x = u, \ y = u - v \). Sketch the image \( G \) of the region \( R \) in \( uv \)-coordinates.

**Solution:** The region is \( 0 \leq v \leq u - u^2, \quad 0 \leq u \leq 1. \)

(c) Use the substitution principle
\[ \int \int_{R} f(x,y) \, dxdy = \int \int_{G} f(u, u - v)J(u, v) \, dudv \]
to express \( \int \int_{R} \sin(xy) \, dxdy \) as an iterated integral in \( uv \)-coordinates.

**Solution:** \( J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = -1 \), so the answer is:
\[ \int_{0}^{1} \int_{u-u^2}^{u} \sin(u^2 - uv) \, dv \, du. \]

Note: Since we always iterate our integrals in the direction of increasing coordinate, we must take the absolute value \( |J| = |-1| = 1 \) in the iterated integral.