Name: Solutions

Student ID#: ______________

Section: ______________

# Midterm Exam 1

Friday, January 31

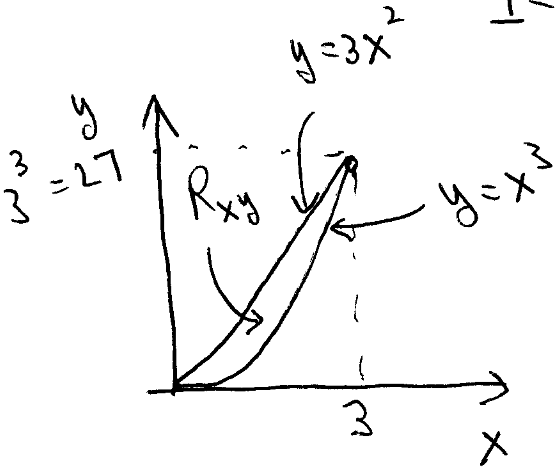
MAT 21D, Temple, Winter 2014

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

| Problem | Your Score | Maximum Score |
|---|---|---|
| 1 | | 20 |
| 2 | | 20 |
| 3 | | 20 |
| 4 | | 20 |
| 5 | | 20 |
| Total | | 100 |

**Problem #1 (20pts): (a)** Sketch the region of integration $\mathbf{R}_{xy}$ determined by the iterated integral

$$I = \int_0^3 \int_{x^3}^{3x^2} \cot(xy)\, dy dx. \tag{1}$$




$$y = x^3 \Leftrightarrow \sqrt[3]{y} = x$$

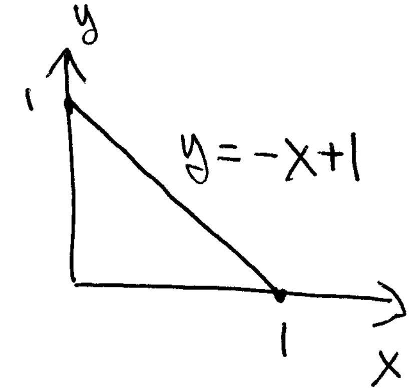
$$y = 3x^2 \Leftrightarrow \sqrt{\frac{y}{3}} = x$$

$$I = \int_0^{27} \int_{\sqrt{y/3}}^{\sqrt[3]{y}} \cot(xy)\, dx$$

**(b)** Rewrite (1) with order of integration reversed. (Do not evaluate)

**Problem #2 (20pts):** Consider a triangular metal plate with three corners (00), (1,0), (0,1) meters and constant density $\delta(x,y) = 3\ kg/m^2$.
**(a)** Set up an *iterated* integral for the mass of the plate, and give the dimensional units of the answer. (No need to evaluate the integral, but *put in units.*)

$$\text{Mass} = \int_0^1 \int_0^{-x+1} 3\, dy\, dx \quad \underline{\underline{kg}}$$




**(b)** Give the center of mass $(\bar{x}, \bar{y})$ in terms of *iterated* integrals. Give the resulting dimensional units of the answer. (Again, no need to evaluate the integrals. Note by symmetry $\bar{x} = \bar{y}$.)

$$\bar{x} = \bar{y} = \int_0^1 \int_0^{-x+1} x \cdot 3\, dy\, dx = \frac{M_y}{M} = \frac{M_x}{M} \quad \underline{\text{meters}}$$

**(c)** Give the KE stored in rotating the plate about the $y$-axis at $\omega = 3$ radians/sec. in terms of *iterated* integrals. Give the resulting dimensional units of the answer. (Again, no need to solve the integrals.)

$$KE = \frac{1}{2} I_y w^2 = \frac{9}{2} I_y \quad \frac{kg\, m}{s^2}$$

$$I_y = \int_0^1 \int_0^{-x+1} x^2 \cdot 3\, dy\, dx$$

**Problem #3 (20pts):** Integrate in spherical coordinates $(\rho, \phi, \theta)$ to find the volume of the region cut from the sphere $x^2 + y^2 + z^2 = 4$, lying between the cones $\phi = \pi/6$ and $\phi = \pi/4$.

Region: $0 \leq \theta \leq 2\pi$

$\pi/6 \leq \varphi \leq \pi/4$

$0 \leq \rho \leq 2$

$$V = \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/4} \sin\varphi \, d\varphi \, d\theta$$

$$\left.\frac{\rho^3}{3}\right]_0^2$$

$$= \frac{8}{3}\cdot 2\pi \int_{\pi/6}^{\pi/4} \sin\varphi \, d\varphi = \frac{16\pi}{3}\left(-\cos\varphi\right)\Big]_{\pi/6}^{\pi/4}$$

$$\frac{16\pi}{3}\left(\cos\tfrac{\pi}{6} - \cos\tfrac{\pi}{4}\right) = \frac{8\pi}{3}\left(\sqrt{3} - \sqrt{2}\right)$$

$$\frac{\sqrt{3}}{2} \quad - \quad \frac{\sqrt{2}}{2}$$

**Problem #4 (20pts):** The mapping $x = u + v$, $y = 2v$, $z = 3w$ takes a domain $D_{xyz}$ in $(x, y, z)$-space to the 3-D rectangular region $0 \leq u \leq 1$, $0 \leq v \leq 2$, $0 \leq w \leq 1$. Evaluate

$$I = \iiint_{D_{xyz}} \left( \frac{2x - y}{2} + \frac{z}{3} \right) dx dy dz.$$

$$I = \int_0^1 \int_0^2 \int_0^1 \frac{2(u+\cancel{v}) - \cancel{2v}}{2} + w J \, du \, dv \, dw$$

$J=6$

$$= \int_0^1 \int_0^2 \left. \frac{u^2}{2} + wu \right]_{u=0}^{u=1} J \, dv \, dw = \int_0^1 \int_0^2 \frac{1}{2} + w J \, dv \, dw$$

$$= \int_0^1 \left[ \frac{1}{2} v + wv \right]_{v=0}^{v=2} J \, dw = \int_0^1 1 + 2w J \, dw$$

$$= 6 \left. w + w^2 \right]_0^1 = 1 + 1^{6} = \boxed{12}$$

**Problem #5 (20pts):**
Use polar coordinates (and the trick in class) to evaluate $I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$, the area under the Gaussian.

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\,dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right] e^{-y^2} dy = \left[\int_{-\infty}^{\infty} e^{-x^2} dx\right]^2$$

$x = r\cos\theta$

$y = r\sin\theta$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq \infty$

$$= \int_0^{2\pi}\int_0^{\infty} r e^{-r^2}\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} r e^{-r^2}\,dr$$

$u = r^2$

$du = 2r\,dr$

$$= 2\pi \cdot \frac{1}{2}\int_0^{\infty} e^{-u}\,du = -\pi e^{-u}\Big]_0^{\infty} = \pi(-0+1)$$

$$= \pi. \quad \therefore \boxed{I = \sqrt{\pi}}$$