MIDTERM EXAM II–Solutions
Math 21D
Temple-F06

Write solutions on the paper provided. Put your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

Problem 1. (20pts) (a) Calculate the gradient $\nabla f(x,y,z)$:

$$f(x,y,z) = sin(xy^2z^3)$$

Soln: $\nabla f = y^2z^3 sin(xy^2z^3)\mathbf{i} + 2xyz^3 sin(xy^2z^3)\mathbf{j} + 3xy^2z^2 sin(xy^2z^3)\mathbf{k}$.

(b) Calculate $Curl\ \mathbf{F}(x,y,z) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{bmatrix}$ when

$$\mathbf{F}(x,y,z) = xyz^3\mathbf{i} + xz\mathbf{j} + z^{10}\mathbf{k}.$$

Soln:

$$Curl\mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^3 & xz & z^{10} \end{bmatrix} = \mathbf{i}(0-x) - \mathbf{j}(0-3xyz^2) + \mathbf{k}(z-xz^3)$$

(c) Calculate the divergence $div\ \mathbf{F}(x,y,z) = M_x + N_y + P_z$:

$$\mathbf{F}(x,y,z) = xyz^3\mathbf{i} + xz\mathbf{j} + z^{10}\mathbf{k}$$

Soln: $div\ \mathbf{F} = yz^3 + 0 + 10z^9$

(d) Label the arrows with div, ∇ and $Curl$ and the (?)'s with correct dimensions of the space on which these operators act, ordered so that two in a row make zero.

$$\mathbf{R}^? \to \mathbf{R}^? \to \mathbf{R}^? \to \mathbf{R}^?$$

Soln: $\mathbf{R}^1 \to \nabla \to \mathbf{R}^3 \to Curl \to \mathbf{R}^3 \to div \to \mathbf{R}^1$

Problem 2. (20pts) Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\, ds$, where $\mathcal{C}$ is the straight line from $P = (-1, -1, 0)$ to $Q = (1, 0, 1)$, and $\mathbf{F} = x\mathbf{i} + 3\mathbf{j} - y\mathbf{k}$.

Soln: $\mathbf{x}(t) = P + t(Q - P) = (-1 + 2t, -1 + t, t)$ so $dx = 2dt, dy = dt, dz = dt$. Thus

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\ ds &= \int_{\mathcal{C}} xdx + 3dy - ydz = \int_0^1 (-1+2t)2dt + 3dt - (-1+t)dt \\ &= \int_0^1 (2+3t)dt = 2t + \frac{3}{2}t^2]_0^1 = 2 + \frac{3}{2} = 7/2.\end{aligned}$$

Problem 3. (20pts) Let R denote the box $0 \le x \le 1$, $0 \le y \le 1$ in the xy-plane, and let $\mathcal{C}$ denote the boundary of R oriented counterclockwise. Assume that a fluid of density $\delta = y\ kg/m^2$ and velocity $v = x\mathbf{i} + y\mathbf{j}\ m/s$ is flowing through the box R.

(a) Find the mass flux vector $\mathbf{F} = \delta\mathbf{v}$, and determine its dimensional units.

Soln: $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ has dimensions of $\frac{kg}{m\ s}$.

(b) The line integral $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\ ds$ gives the rate at which mass passes outward through $\mathcal{C}$. Write this as the integral of a 1-form using the components of F. (Do not evaluate it.)

Soln: $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\, ds = \int_{\mathcal{C}}(M,N)\cdot(T_y, -T_x)\ ds = \int_{\mathcal{C}}(M,N)\cdot(dy, -dx) = \int_{\mathcal{C}} Mdy - Ndx = \int_{\mathcal{C}} xydy - y^2dx$

(c) State the divergence theorem, and use this to determine the rate at which mass is leaving the box, by evaluating a double integral over the box.

Soln: $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\ ds = \int\int_R div\ \mathbf{F}\ dA = \int\int_R M_x + N_y\ dA = \int_0^1 \int_0^1 y + 2y\ dydx = \int_0^1 \frac{3}{2}y^2]_0^1 dx = 3/2$.

Problem 4. (20pts) Let $\mathbf{F}(x, y) = y\sin(xy)\mathbf{i} + (y + x\sin(xy))\mathbf{j}$

(a) Evaluate $Curl\mathbf{F}$ state the correct theorem to determine whether F is conservative.

Soln: $Curl\mathbf{F} = (N_x - M_y)\mathbf{k} = (\sin(xy) + xy\cos(xy) - \sin(xy) - xy\cos(xy))\mathbf{k} = 0$. The theorem says that if $CurlF = 0$ in a simply connected domain, then $\mathbf{F}$ is conservative.

(b) Use the standard procedure for finding f such that $\mathbf{F} = \nabla f$.

Soln: $f(x, y) = \int_x y\sin(xy)dx = -\cos(xy) + g(y)$. Differentiating this wrt y and setting it equal to N gives $x\sin(xy) + g'(y) = x\sin(xy) + y$ which implies that $g(y) = \frac{y^2}{2}$. So $f(x, y) = -\cos(xy) + \frac{y^2}{2}$.

(c) Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\ ds$ where $\mathcal{C}$ is a curve that takes $P = (\pi/2, 0)$ to $Q = (0, 1)$.

Soln: $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\ ds = f(0, 1) - f(\pi/2, 0) = \sin(0) + 1/2 - \sin(0) - 0 = 1/2$.

Problem 5. (20pts) Prove that if $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} = \nabla f(x,y)$, then $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\ ds = f(Q) - f(P)$ for any smooth curve taking P to Q. (You may assume F is smooth everywhere.)

Soln: Let $\mathbf{x}(t) = (x(t), y(t))$, $a \leq t \leq b$ be any smooth parameterization of $\mathcal{C}$ so that $\mathbf{x}(a) = P$ and $\mathbf{x}(b) = Q$. Then we can write: $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\ ds = \int_a^b \mathbf{F} \cdot \mathbf{v}(t)\ dt = \int_a^b \left\{ \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right\} dt = \int_a^b \frac{d}{dt} f(x(t), y(t))dt = f(x(b), y(b)) - f(x(a), y(a)) = f(Q) - f(P)$.