Name: Solutions

Student ID#: ______________________

Section: ______________________

Midterm Exam 2

Monday March 3

MAT 21D, Temple, Winter 2014

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): A particle of mass $m = 2$ moves along a trajectory given by

$$\mathbf{r}(t) = a\cos bt\ \mathbf{i} + a\sin bt\ \mathbf{j} + ct\ \mathbf{k},$$

where t is in seconds, $\mathbf{r}$ is in km, and a, b, c are dimensional constants. Find formulas for the following in terms of a, b, c:

(a) The velocity vector $\mathbf{v}(t) = r'(t) = -ab\sin bt\ \underset{\sim}{i} + ab\cos bt\ \underset{\sim}{j} + c\underset{\sim}{k}$

(b) The speed $\frac{ds}{dt} = v(t) = \|\vec{v}\| = \sqrt{a^2b^2(\sin^2 t + \cos^2 t) + c^2}$

$$= \sqrt{a^2b^2 + c^2}$$

(c) The acceleration vector $\mathbf{a}(t) = \vec{v}'(t) = -ab^2\cos bt - ab^2\sin bt$

(d) The unit tangent vector $\mathbf{T}(t) = \dfrac{\vec{v}}{\|\vec{v}\|}$

$$= \frac{-ab}{\sqrt{a^2b^2+c^2}}\sin bt\ \underset{\sim}{i} + \frac{ab}{\sqrt{a^2b^2+c^2}}\cos bt\ \underset{\sim}{j} + \frac{c}{\sqrt{a^2b^2+c^2}}\ \underset{\sim}{k}$$

(e) The unit normal $\mathbf{N}(t) = \dfrac{d\vec{T}}{dt} \Big/ \left\| \dfrac{d\vec{T}}{dt} \right\|$

$$\frac{d\vec{T}}{dt} = -\frac{ab^2 \cos bt}{\sqrt{a^2b^2+c^2}}\,\underset{\sim}{i} + \frac{ab^2 \sin bt}{\sqrt{a^2b^2+c^2}}\,\underset{\sim}{j}, \quad \left\|\frac{d\vec{T}}{dt}\right\| = \frac{ab^2}{\sqrt{a^2b^2+c^2}}$$

$$\vec{N} = -\cos bt\,\underset{\sim}{i} + \sin bt\,\underset{\sim}{j}$$

(f) The curvature $\kappa(t)$. $= \dfrac{1}{\|\vec{V}\|} \left\| \dfrac{d\vec{T}}{dt} \right\| = \dfrac{1}{\sqrt{a^2b^2+c^2}} \dfrac{ab^2}{\sqrt{a^2b^2+c^2}}$

$$= \frac{ab^2}{a^2b^2+c^2}$$

(g) The length of the component of $\mathbf{a}(t)$ in direction of $\mathbf{N}$.

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \underbrace{\kappa\|\vec{V}\|^2}_{a_N}\vec{N}$$

$$a_N = \kappa\|\vec{V}\|^2 = \frac{ab^2}{a^2b^2+c^2} \cdot \left(\sqrt{a^2b^2+c^2}\right)^2 = ab^2$$

Problem #2 (20pts): Let C denote the unit circle oriented counter-clockwise, and let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

(a) Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ directly by parameterization.

$\vec{r}(t) = \cos t \, \underset{\sim}{i} + \sin t \, \underset{\sim}{j} \qquad \vec{r}\,'(t) = -\sin t \, \underset{\sim}{i} + \cos t \, \underset{\sim}{j}$

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_0^{2\pi} \vec{F} \cdot \vec{v} \, dt = \int_0^{2\pi} \overrightarrow{(\sin t, -\cos t)} \cdot \overrightarrow{(-\sin t, \cos t)} \, dt$$

$$= \int_0^{2\pi} -\sin^2 t - \cos^2 t \, dt = -2\pi$$

(b) Evaluate $\int_C \mathbf{F} \cdot \mathbf{T} ds$ by Green's Theorem.

$$\int_C \vec{F} \cdot \vec{T} \, ds = \iint_A \underbrace{N_x - M_y}_{\substack{N = -x \\ M = y}} \, dx dy = \iint_{\substack{\text{unit} \\ \text{circle}}} -1 - 1 \, dx dy$$

unit circle

$$= \pi \cdot 1^2 (-2) = -2\pi$$

Problem #3 (20pts): Find the following assuming

$$\mathbf{F}(x, y, z) = 2xyz\mathbf{i} + (x^2 z + 2y)\mathbf{j} + (x^2 y + 1)\mathbf{k}.$$

(a) Div $\mathbf{F}$ $= M_x + N_y + P_z = 2yz + 2 + 0 = 2(yz+1)$

(b) Curl $\mathbf{F}$ $= \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & x^2z+2y & x^2y+1 \end{vmatrix}$

$= \underset{\sim}{i}(x^2 - x^2) - \underset{\sim}{j}(2xy - 2xy) + \underset{\sim}{k}(2xz - 2xz) = 0$

(c) Find f such that $\mathbf{F} = \nabla\ f$

$$f(x,y,z) = \int_x M = \int_x 2xyz\,dx = \frac{2x^2yz}{2} + g(y,z)$$

$$\frac{\partial f}{\partial y} = x^2z + \frac{\partial g}{\partial y} = N = x^2z + 2y \Rightarrow \frac{\partial g}{\partial y} = 2y$$

$$= g(y,z) = y^2 + h(z)$$

$$\frac{\partial f}{\partial z} = x^2y + h'(z) = P = x^2y + 1 \Rightarrow h'(z) = 1 \Rightarrow h(z) = z$$

$$f(x,y,z) = x^2yz + y^2 + z$$

(d) Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ along any smooth curve C taking $A = (-1, 1, 2)$ to $B = (1, 1, -1)$.

$$\int_C \vec{F} \cdot \vec{T}\,ds = f(B) - f(A) = f(1,1,-1) - f(-1,1,2)$$

$$= \left[(1)^2(1)(-1) + (1)^2 + (-1)\right] - \left[(-1)^2(1)(2) + (1)^2 + 2\right]$$

$$= \cancel{-1} + \cancel{1} - 1 - 2 - 1 - 2 = -6$$

Problem #4 (20pts): (a) Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field, where M, N, P are assumed to be given functions of (x, y, z). Use Leibniz's substitution principle to show the following are equal: (Here $\mathbf{r}(t)$ denotes any parameterization of curve C.)

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_C M dx + N dy + P dz.$$

Given $\vec{r}(t)$, $d\vec{r} = \vec{v}(t)\,dt$, $ds = \|\vec{v}\|dt$,

$\vec{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$ so $\vec{v} = \|\vec{v}\|\vec{T} = \dfrac{ds}{dt}\vec{T} \Rightarrow \vec{v}\,dt = \vec{T}\,ds$

$\therefore \int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\,\vec{v}\,dt$

Since $\dfrac{d\vec{r}}{dt} = \vec{v}$, $d\vec{r} = \vec{v}\,dt = \left(\dfrac{dx}{dt}\underset{\sim}{i} + \dfrac{dy}{dt}\underset{\sim}{j} + \dfrac{dz}{dt}\underset{\sim}{k}\right)dt$

$= dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}$

$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\overrightarrow{(dx, dy, dz)} = \int_C M dx + N dy + P dz.$

(b) Assume further that $F = -\nabla P$ for some scalar function $P(x, y, z)$, and that $\mathbf{F} = m\mathbf{a}$ is the total force creating the motion of a particle along a curve $\mathbf{r}(t)$ between two points of motion $\mathbf{r}(t_a) = A$ and $\mathbf{r}(t_b) = B$. State and prove the principle of Conservation of Energy. Give complete arguments. (Hint: evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ two different ways.)

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_{t_a}^{t_b} \vec{F} \cdot \vec{v}\, dt = \int_{t_a}^{t_b} \nabla f \cdot \vec{v}\, dt$$

$$= \int_{t_a}^{t_b} \frac{d}{dt} f(\vec{r}(t))\, dt = \vec{r}(t_b) - \vec{r}(t_a)$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C m\vec{a} \cdot \vec{T}\, ds = \int_{t_a}^{t_b} m\vec{a} \cdot \vec{v}\, dt$$

$$= \int_{t_a}^{t_b} \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v})\, dt = \frac{1}{2} m \|\vec{v}(t_b)\|^2 - \frac{1}{2} m \|\vec{v}(t_a)\|^2$$

Problem #5 (20pts): Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be the trajectory of a planet moving in a plane with the Sun at the center $(x, y) = 0$. In the Newton-Kepler problem we showed that if $x = r\cos\theta$ and $y = r\sin\theta$, then differentiating $\mathbf{r}(t)$ twice and simplifying (using equal area in equal time) led to

$$\begin{aligned} \ddot{x}\cos\theta + \ddot{y}\sin\theta &= \ddot{r} - \frac{H^2}{r^3} && (1) \\ -\ddot{x}\sin\theta + \ddot{y}\cos\theta &= 0. && (2) \end{aligned}$$

Use these to solve for the acceleration vector $\mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$, and show that the acceleration vector points in the direction of the position vector $\mathbf{r}(t)$, (i.e., toward the sun).

mult by cos/sin $\quad \ddot{x}\cos^2\theta + \ddot{y}\cos\theta\sin\theta = \left(\ddot{r} - \frac{H^2}{r^3}\right)\cos\theta$

subt $\quad -\ddot{x}\sin^2\theta + \ddot{y}\cos\theta\sin\theta = 0$

$$\ddot{x} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\cos\theta$$

mult by sin/cos $\quad \ddot{x}\cos\theta\sin\theta + \ddot{y}\sin^2\theta = \left(\ddot{r} - \frac{H^2}{r^3}\right)\sin\theta$

add $\quad -\ddot{x}\sin\theta\cos\theta + \ddot{y}\cos^2\theta = 0$

$$\ddot{y} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\sin\theta$$

$$\vec{a} = \overrightarrow{(\ddot{x}, \ddot{y})} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\overrightarrow{(\cos\theta, \sin\theta)} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\left(\frac{1}{r}\right)\overrightarrow{(x, y)}$$

points in line with $\vec{r}$ $\quad = c(t)\vec{r}(t)$