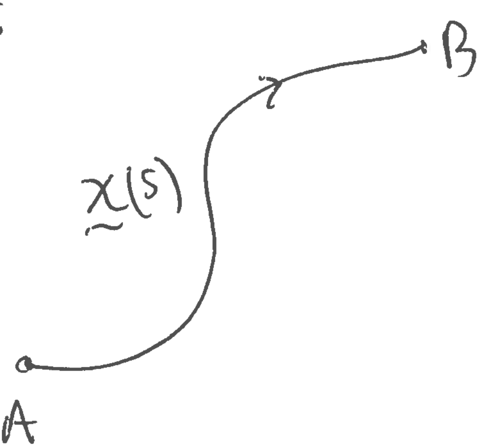
Conservation of Energy

- Line integral:

$$\int_C \vec F\cdot\vec T\,ds = \int_C \vec F\cdot d\vec r = \int_C \vec F\cdot\underbrace{\vec v\,dt}_{d\vec r} = \int_C M\,dx + N\,dy + P\,dz$$

$$\vec T\,ds = d\vec r = \vec v\,dt$$

$$\int_{s_a}^{s_b} \vec F(\underset{\sim}{x}(s))\,\vec T(\underset{\sim}{x}(s))\,ds$$



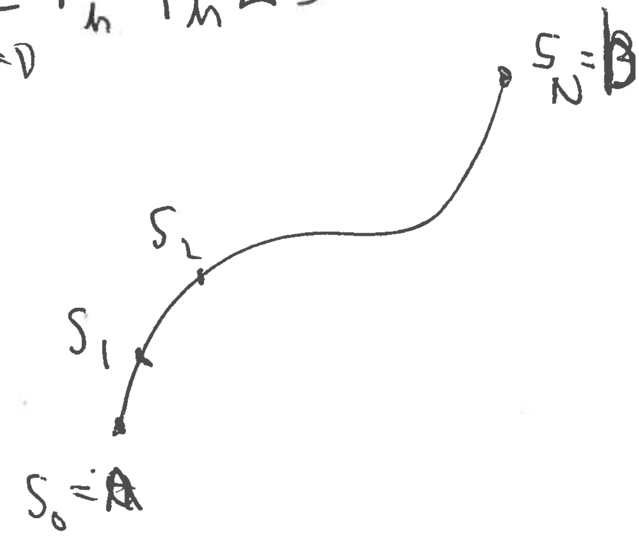
$$\vec v = \frac{ds}{dt}\vec T$$

$$\vec T\,ds = \vec v\,dt = \overrightarrow{\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)}dt = \overrightarrow{(dx,dy,dz)}$$

$$= d\vec r$$

$$\vec F = \overrightarrow{(M,N,P)} \qquad \vec F\cdot d\vec r = M\,dx + N\,dy + P\,dz$$

- $\int_C \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty} \sum_{k=0}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s$



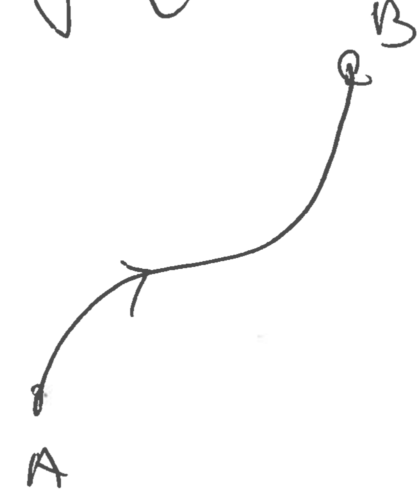
"~~A~~ sum of weighted length, the wt being the comp of $\vec{F}$ tangent to C"

~~Phy~~ Math: $\int_C \vec{F}\cdot\vec{T}\,ds$ = line integral of vector field $\vec{F}$

$\oint_C \vec{F}\cdot\vec{T}\,ds$ = circulation in $\vec{F}$ around C

Physics:

$\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$ = "Work done by force $\vec{F}$ in motion along $\mathcal{C}$"



Theorem: If a particle mass moves along $\mathcal{C}$ and $\vec{F}$ is the only force acting —

— so $\vec{F} = m\vec{a}$ —

Then

$$\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \Delta KE = \frac{1}{2}m|\vec{V}_b|^2 - \frac{1}{2}m|\vec{V}_a|^2$$

Proof: 1-D case first:

$$\int_{s_a}^{s_b} \vec{F}\, ds \underset{\substack{\uparrow \\ \vec{F} = m\vec{a}}}{=} \int_{s_a}^{s_b} m \frac{d^2x}{dt^2}\, ds \underset{\substack{\uparrow \\ ds = v\,dt}}{=} \int_{t_a}^{t_b} m \frac{d^2x}{dt^2} \cdot \underbrace{\frac{dx}{dt}\, dt}_{v\,dt}$$

$$= m \int_{t_a}^{t_b} \underbrace{\frac{1}{2} \frac{d}{dt} \left(\frac{dx}{dt}\right)^2}_{2 \frac{dx}{dt} \frac{d^2x}{dt^2} \checkmark} = m \frac{1}{2} v^2 \Big]_{t_a}^{t_b}$$

$$= \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2$$

~~General case~~

General Case —

$$\int_e \vec{F}\cdot\vec{T}\,ds = \int_{t_a}^{t_b} \vec{F}\cdot\vec{v}\,dt = \int_{t_a}^{t_b} m\underbrace{\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt}}_{\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right)}dt$$

$\vec{F} = m\vec{a}$

dot product

<u>check</u>: $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt}\frac{d^2\vec{r}}{dt^2}$

dot product

⇒ Liebniz prod rule holds

$= 2\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt}$ ✓

$$= \int_{t_a}^{t_b} \frac{d}{dt}\left(\frac{1}{2}m\|\vec{v}\|^2\right)dt = \frac{1}{2}m\|\vec{v}_b\| - \frac{1}{2}m\|\vec{v}_a\|$$

<u>Conclude</u>: If $\vec{F}$ creates the motion (so $\vec{F} = m\vec{a}$) then

$$\int_C \vec{F} \cdot \vec{T}\, ds = \Delta KE \checkmark$$

$\boxed{\text{B}}$ Conservative Vector Fields —

We say $\vec{F}$ is <u>conservative</u> if

$$\vec{F} = \nabla f$$

(Very Special)

- We have: $\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3} \Rightarrow$ Newtons Grav. Force is conservative!

Theorem: If $\vec{F} = \nabla f$, then

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = f(B) - f(A)$$

Proof: Assume $\vec{F} = \nabla f$. Write

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = \int_{t_A}^{t_B} \vec{F} \cdot \vec{v}\, dt = \int_{t_A}^{t_B} \nabla f \cdot \frac{d\vec{r}}{dt}\, dt$$

assume any parametre $\vec{r}(t)$ $t_A \le t \le t_B$

$$= \int_{t_A}^{t_B} \underbrace{\nabla f(\vec{r}(t)) \frac{d\vec{r}}{dt}}\, dt$$

$$\frac{d}{dt} f(\vec{r}(t)) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z}$$

chn rule $= \nabla f \cdot \vec{v}$

Thus:

$$\int_{t_a}^{t_b} \nabla f \cdot \vec{v}\, dt = \int_{t_a}^{t_b} \frac{d}{dt}\left(f(\vec{r}(t))\right) dt$$

$$= f(\vec{r}(t_B)) - f(\vec{r}(t_A))$$

$$= f(B) - f(A)$$

Conclude:

① $\displaystyle\int_{e} \vec{F} \cdot \vec{T}\, ds \underset{\vec{F} = m\vec{a}}{\overset{\uparrow}{=}} \frac{1}{2}m\|\vec{v}_b\|^2 - \frac{1}{2}m\|\vec{v}_a\|^2$

② $\displaystyle\int_{e} \vec{F} \cdot \vec{T}\, ds \underset{\vec{F} = \nabla f}{\overset{\uparrow}{=}} f(B) - f(A)$

Conclude: If both $\vec{F} = \nabla f$ and $\vec{F} = m\vec{a}$ then

$$\Delta f = \int_C \vec{F} \cdot \vec{T}\, ds = \Delta\left(\tfrac{1}{2} m \|v\|^2\right)$$

In Physics, we say $\vec{F}$ is conservative if $\vec{F} = -\nabla P = \nabla f$ so

P is the potential energy

$$\Delta PE = P(B) - P(A)$$

$$\Delta KE = \tfrac{1}{2} m \|\vec{v}_B\|^2 - \tfrac{1}{2} m \|\vec{v}_A\|^2$$

$$\therefore \quad \Delta KE + \Delta PE = \cancel{\text{constant}}\ 0$$

This is the principle of conservation of energy in physics.

Eg: Newton gravity: (Motion of planet around Sun

$$\underset{M_p \vec{a}}{\vec{F}} = -\hat{G}\,\frac{\vec{r}}{r^3} = \hat{G}\,\nabla\left(\frac{1}{r}\right) \qquad \hat{G} = GM_sM_E)$$

Then

$$= -\hat{G}\,\nabla P$$

$$P = -\frac{1}{r}$$

Conclude: (Energy is conserved).

$$\underbrace{\frac{1}{2}M_p\|\vec{V}_B\|^2 - \frac{1}{2}M_p\|\vec{V}_A\|^2}_{\Delta KE} + \underbrace{\left(-\frac{1}{r_B} + \frac{1}{r_A}\right)}_{\Delta PE} = 0$$

Defn: $E(t) = KE + PE = \frac{1}{2}M\|V\|^2 + \left(-\frac{1}{r}\right)$

Then $E(t)$ is constant along planetary orbits —

<u>Newton's Law of Gravit</u>

Explained the motion of the planets & created the principle of conservation of energy!