Recall 2 forms of Green’s Theorem $R^2$:

1. Line element form
   \[ \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_R \left( N_x - M_y \right) dx \, dy \]

   - Counter-clockwise
   - Outer normal

2. Flux form
   \[ \iint_R \nabla \times \mathbf{F} \cdot d\mathbf{A} = \iint_R \nabla \cdot \mathbf{F} \, dxdy \]
   \[ \mathbf{F} = M \mathbf{i} + N \mathbf{j} \]

- They generalize to Stokes Thm & Divergence Thm respectively.
Stokes Theorem $\mathbb{R}^3$

\[ \oint F \cdot ds = \iint \text{Curl} F \cdot \hat{n} \, d\sigma \]

(component of \text{Curl} F in direction of \hat{n})

\[
\text{Curl} F = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\partial_x & \partial_y & \partial_z \\
M & N & P
\end{vmatrix}
\]

Note: $\iint F \cdot \hat{n} \, d\sigma = "\text{Flux of } F \text{ thru } D"$

\[ \iint \text{Curl} F \cdot \hat{n} \, d\sigma = "\text{Flux of the } \text{curl} \text{ of } F \text{ thru } D" \\
\]

"Line integral of $F$ around $\Gamma$ - curl $F$ = flux of the curl thru surface"
2) Divergence Theorem $\mathbf{R}^3$—

Given 3-d volume $V$ bounded by a closed surface $A$

\[ \iiint_V \nabla \cdot \mathbf{F} \, dV = \iint_A \mathbf{F} \cdot \mathbf{n} \, dA \]

"The flux of $\mathbf{F}$ thru $A$ equals the flux of $\mathbf{F}$ out thru boundary $A$"

\[ \text{div } \mathbf{F} = M_x + M_y + N_z \]
Q: What is the flux?

Ans: \( \iint F \cdot \hat{n} \, d\sigma \) = rate at which stuff passes thru \( \sigma \)

Ex: Simplest way to see this - assume a density \( \rho(x, t) \) is flowing with velocity \( \vec{V}(x, t) \) \( x = (x, y, z) \)

\[ \rho = \frac{\text{mass}}{\text{vol}} \]

\[ \vec{V} = (V_x, V_y, V_z)(x, t) = \text{velocity} = \frac{\text{distance}}{\text{time}} \]

Def: \( \vec{F} = \rho \vec{V} \) = mass flux vector
More generally — any time a density of stuff is flowing you have a stuff flux vector \( \mathbf{E} \).

\[ \mathbf{E} = \frac{\text{stuff}}{\text{volume}} \] (E.g. charge, chemical energy, moment)

\[ \mathbf{V} = (V_x, V_y, V_z) \] velocity at which it's flowing

\[ \mathbf{F} = \mathbf{E} \cdot \mathbf{V} = \text{stuff flux vector} \]

Claim: \[ \iint \mathbf{F} \cdot \hat{n} \, dS = \text{rate at which stuff is passing out through } \mathcal{A} \] in direction \( \hat{v} \)
To see this:

\[
(S\cdot\vec{v}) = \frac{\text{mass}}{\text{vol}} \cdot \frac{\text{dist}}{\text{time}} = \frac{\text{mass}}{\text{area} \cdot \text{time}}
\]

\[
\vec{p}\cdot\vec{n} = \frac{\text{mass}}{\text{area} \cdot \text{time}} \quad \text{moving normal to do-}
\]

\[
\vec{p}\cdot\vec{n} \cdot \text{do} = \frac{\text{mass}}{\text{time}} \quad \text{pass thru do-}
\]

\[
\frac{\text{mass}}{\text{area}} \quad \frac{\text{area}}{\text{time}}
\]

\[
\int \int \vec{p}\cdot\vec{n} \cdot \text{do} \approx \sum \vec{p}\cdot\vec{n}\cdot\text{do}
\]

\[
\frac{\text{mass}}{\text{time}} \quad \frac{\text{mass}}{\text{time}}
\]

\[
\int \text{mass thru do-}
\]

\[
\int \text{mass thru do-}
\]
Eq. The divergence thm states — "the rate at which mass fluxes thru boundary surface $\mathcal{S}$ equals the integral of div $\mathbf{F}$ over $\mathcal{V}$"

$$\iiint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, d\mathcal{S} = \iiint_{\mathcal{V}} \text{div} \mathbf{F} \, d\mathcal{V}$$

Application: Find the equation the expresses "mass is conserved" [Continuity eqn]
Soln: \( \mathbf{F} \cdot \mathbf{n} = \text{mass flux vector} \)

Choose any 3-d region \( \mathcal{V} \) with bounding surface \( \partial \mathcal{V} \)

\[
\iiint_{\partial \mathcal{V}} \mathbf{F} \cdot \mathbf{n} \, d\mathbf{S} = \text{rate at which mass passes out of } \mathcal{V} \text{ thru } \partial \mathcal{V}
\]

\[
= \iiint_{\mathcal{V}} \text{div } \mathbf{F} \, d\mathbf{V}
\]

Conservation of mass requires that this exactly equal the rate at which the total mass in \( \mathcal{V} \) is changing —
That is

\[ M = \iiint p \, dv = \sum p_i \Delta v_i \approx \text{total mass} \]

\[ \Delta \text{mass} \cdot \frac{1}{\Delta t} \]

\[ \frac{dM}{dt} = \frac{d}{dt} \iiint p(x, t) \, dx \, dy \, dz \]

\[ = \iiint \frac{\partial p}{\partial t} (x, t) \, dx \, dy \, dz \]

\[ = \text{rate at which total mass is changing in } V \]
We have - Conservation of Mass -

"The time rate of change of M in \( \Omega \) = rate at which mass is fluxing out thru boundary of \( \Omega \)

\[
\frac{d}{dt} M = -\iint_{\partial \Omega} \mathbf{F} \cdot \mathbf{n} \, d\sigma
\]

\[
\iiint_{\Omega} \mathbf{F}_t \, dv = -\iiint_{\Omega} \text{div} \mathbf{F} \, dv
\]

\[
\iiint_{\Omega} \left[ \mathbf{F}_t + \text{div} \mathbf{F} \right] \, dv = 0
\]
Must hold for every volume $V$ and every time $t$,

$$\rho_t + \text{div } \rho \mathbf{v} = 0$$

Continuity Eqn

Expresses "conservation of mass"