Name: ____________________

Student ID#: ____________________

Section: ____________________

Final Exam

Wednesday March 18, 1-3pm

MAT 21D, Temple, Winter 2015

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (25pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ and evaluate the iterated integral

$$\int_0^1 \int_{x^4}^{x} x^2 y \, dy dx. \tag{1}$$

(b) Rewrite (1) with order of integration reversed. (Do not re-evaluate).

Problem #2 (25pts): Recall uniform circular motion

$$\mathbf{r}(t) = r_0 \cos(\omega t)\mathbf{i} + r_0 \sin(\omega t)\mathbf{j}.$$

(a) Find the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the tangential and normal components a_T and a_N of the acceleration vector $\mathbf{a}$, (so that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$).

(b) Find r_0 and ω in terms of the mass m and velocity $v = ds/dt$ so that Newton's force law holds in the form

$$m\mathbf{a}(t) = -\mathbf{r}(t).$$

Problem #3 (25pts): (**(a)** Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field, where M, N, P are assumed to be given functions of (x, y, z). Use Leibniz's substitution principle to show the following are equal: (Here $\mathbf{r}(t)$ denotes any parameterization of curve C.)

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_C M dx + N dy + P dz.$$

(b) Assume further that $F = -\nabla U$ for some scalar function $U(x, y, z)$, and that $\mathbf{F} = m\mathbf{a}$ is the total force creating the motion of a particle along a curve $\mathbf{r}(t)$ between two points of motion $\mathbf{r}(a) = A$ and $\mathbf{r}(b) = B$. State and prove the principle of Conservation of Energy. Give complete arguments. (Hint: evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ two different ways.)

Problem #4 (25pts): Let

$$\mathbf{F}(x, y, z) = y^2\mathbf{i} + (2xy + z^2)\mathbf{j} + (2yz + 3z^2)\mathbf{k}.$$

(a) Find Curl $\mathbf{F} = \nabla \times \mathbf{F}$.

(b) Find f such that $\mathbf{F} = \nabla f$.

(c) Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ along any smooth curve C taking $A = (1, -1, 2)$ to $B = (-1, 1, -1)$.

(d) T or F: "The line integral of **F** around any closed curve is zero."

Problem #5 (20pts): Let

$$\mathbf{F}(x,y) = \frac{-y}{r^2}\mathbf{i} + \frac{x}{r^2}\mathbf{j}, \quad r = \sqrt{x^2+y^2}.$$

(a) Verify that $Curl\mathbf{F} = 0$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ by parameterization, where $\mathcal{C}$ is the circle of center $(0,0)$ and radius $r = 1$.

(c) Recall Stoke's Theorem when $\mathcal{A}$ is the disk $r \leq 1$ reads

$$\int\int_{\mathcal{A}} Curl\mathbf{F} \cdot \mathbf{k} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds.$$

Does it hold in this case? Explain.

Problem #6 (25pts): (a) State Stokes Theorem for a disk $D_\epsilon(P)$ of radius ϵ and center $P = (x_0, y_0, z_0)$ and normal $\mathbf{n}$, and use this to derive the meaning of $Curl\mathbf{F} \cdot \mathbf{n}$ at P as *the circulation in* $\mathbf{F}$ *per area around axis* $\mathbf{n}$ *at* P.

(b) Use (**a**) to show that the maximal circulation per area at P occurs around an axis parallel to the $Curl\mathbf{F}$, and that the length of $Curl\mathbf{F}$ is that maximal circulation per area.

(c) Let $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} + xy\mathbf{k}$. Find the axis $\mathbf{n}$ of maximal circulation at $P = (-1, 1, 2)$.

(c) Find the circulation per area around the axis $\mathbf{j} - \mathbf{k}$ at the point $P_0 = (-1, 1, 2)$.

Problem #7 (25pts): Let

$$\mathbf{r}(u,v) = u^2\mathbf{i} + uv\mathbf{j} + v^2\mathbf{k}, \quad u^2 + v^2 \leq 1,$$

be coordinates for a 2-dimensional surface $\mathcal{S}$ in $\mathcal{R}^3$, so $\mathcal{R}_{uv}$ is $u^2 + v^2 \leq 1$.

(a) Find the unit normal $\mathbf{n}$ to the surface for each $(u,v) \in \mathcal{R}_{uv}$.

–Find the amplification factor $\alpha(u,v)$ so that $d\sigma = \alpha(u,v)dudv$.

–Find a parameterization of the coordinate boundary $u^2 + v^2 = 1$ oriented counter-clockwise.

(b) Evaluate the flux integral $\int\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, d\sigma$ where $\mathbf{F} = \mathbf{i} + \mathbf{k}$.

(c) Let $\mathcal{C}$ denote the curve that surrounds the boundary of $\mathcal{S}$, the image $r(u, v)$ for $u^2+v^2 = 1$, oriented counter-clockwise in the (u, v)-plane. Evaluate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds$ by parameterization, where $\mathbf{F} = -\mathbf{i} + \mathbf{k}$.

Problem #8 (25pts): Recall the Divergence Theorem

$$\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_{\mathcal{V}} Div\mathbf{F}\,dV, \tag{2}$$

where $\mathcal{V}$ is the sphere of radius $\rho = 2$ and $\mathbf{F} = x\mathbf{i} + you\mathbf{j} + z\mathbf{k}$. Verify (2) by explicitly evaluating both sides using spherical coordinates
$x = \rho\cos(\theta)\sin(\phi)$; $\quad y = \rho\sin(\theta)\sin(\phi)$; $\quad z = \rho\cos(\phi)$.
(You may use that on a sphere, $d\sigma = \rho^2 \sin\phi d\phi d\theta$.)