Name: ______________________

Student ID#: ______________________

Section: ______________________

Final Exam–Solutions

Wednesday June 8, 8-10am

MAT 21D, Romik-Temple, Winter 2016

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (25pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ and evaluate the iterated integral

$$\int_0^1 \int_{x^6}^{x} x\, dydx. \tag{1}$$

Solution: $\int_0^1 \int_{x^6}^{x} x\, dydx = \int_0^1 [x - x^6] x\, dx = [\frac{x^3}{3} - \frac{x^8}{8}]_0^1 = \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$

(b) Rewrite (1) with order of integration reversed. (Do not re-evaluate).

Solution: $\int_0^1 \int_{x^6}^{x} x\, dydx = \int_0^1 \int_y^{y^{1/6}} x\, dxdy = \int_0^1 \left[\frac{x^2}{2} |_y^{y^{1/6}} \right] dy = \frac{1}{2} \int_0^1 y^{1/3} - y^2\, dy = \frac{1}{2} \left[\frac{3}{4} y^{4/3} - \frac{1}{3} y^3 \right]_0^1 = \frac{1}{2} \left[\frac{3}{4} - \frac{1}{3} \right] = \frac{5}{24}$

Problem #2 (25pts): Use spherical coordinates (ρ, ϕ, θ) to find the volume of the region $\mathcal{V}$ obtained by removing the cone $\phi \leq \pi/3$ from the sphere $x^2 + y^2 + z^2 = 4$.

Solution:

$V = \int\int\int\int_{\mathcal{V}} dxdydz = \int_0^{2\pi}\int_{\pi/3}^{\pi}\int_0^2 \rho^2 \sin\phi d\rho d\phi d\theta =$
$\int_0^{2\pi}\int_{\pi/3}^{\pi} \frac{\rho^3}{3}|_0^2 \sin\phi d\phi d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/3}^{\pi} \sin\phi d\phi d\theta = \frac{8}{3}\int_0^{2\pi} [-\cos\phi]_{\pi/3}^{\pi} d\theta$
$= \frac{8}{3}\int_0^{2\pi} [1 + 1/2]\, d\theta = \frac{8}{3} \cdot 2\pi \cdot \frac{3}{2} = 8\pi$

Problem #3 (25spts): Find the following assuming

$$\mathbf{F}(x,y,z) = (y^2z + y)\mathbf{i} + (2xyz + x)\mathbf{j} + (xy^2 + 1)\mathbf{k}.$$

(a) Solution:

Div $\mathbf{F} = 2xz$

(b) Solution:

$$\begin{aligned} \text{Curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y^2z + y & 2xyz + x & xy^2 + 1 \end{vmatrix} \\ &= \overrightarrow{(2xy - 2xy, -y^2 + y^2, 2yz + 1 - 2yz - 1)} = 0. \end{aligned}$$

(c) Find f such that $\mathbf{F} = \nabla f$ **Solution:**

$$f(x,y,z) = \int_x M\,dx = \int_x y^2z + y\,dx = y^2zx + yx + g(y,z),$$

so

$$\frac{\partial f}{\partial y} = 2xz + x + g_y = N = 2xyz + x,$$

so $g_y = 0$ and $g = g(z)$. Thus

$$\frac{\partial f}{\partial z} = y^2x + g_z' = P = xy^2 + 1,$$

so $g'(z) = 1$ and $g = z$. Therefore

$$f(x,y,z) = xy^2z + xy + z.$$

(d) Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ along any smooth curve C taking $A = (-1,1,2)$ to $B = (1,1,-1)$.

Solution: By (c), F is conservative so

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= f(1,1,-1) - f(-1,1,2) \\
&= (1)(1)^2(-1) + (1)(1) + (-1) - ((-1)(1)^2(2) - (-1)(1) - 2) \\
&= -1 + 1 - 1 - (-2 - 1 + 2) = -1 - (-1) = 0
\end{aligned}$$

Problem #4 (25pts): Let

$$\mathbf{F}(x, y, z) = \left(\frac{-2y}{x^2 + y^2}\right)\mathbf{i} + \left(\frac{2x}{x^2 + y^2}\right)\mathbf{j} + \mathbf{k}.$$

(a) Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ where C is the unit circle $x^2 + y^2 = 1$ at $z = 1$.

Solution: Let $x = \cos(t)$, $y = \sin(t)$, $z = 1$, so $dx = -\sin(t)dt$, $dy = \cos(t)dt$, $dz = 0$. Thus

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{T}\ ds &= \int_0^{2\pi} M dx + N dy + P dz = \int_0^{2\pi} (-2y)\, dx + (2x)\, dy + (1)\ dz \\
&= \int_0^{2\pi} \{(-2y)\, (-\sin(t)) + (2x)\, (\cos(t)\}\, dt \\
&= \int_0^{2\pi} 2\left\{\sin^2(t) + \cos^2(t)\right\}\, dt = 4\pi
\end{aligned}$$

(b) A calculation would show that Curl $\mathbf{F} = 0$. Explain why $\int_C \mathbf{F} \cdot \mathbf{T}\, ds \neq 0$.

Solution: Because Curl $\mathbf{F}$ is defined only away from the z-axis, the region where Curl $\mathbf{F} = 0$ is not simply connected, so we cannot conclude that $\mathbf{F}$ is conservative.

Problem #5 (25pts): Recall Stokes Theorem: ($\mathcal{S}$ is any surface with boundary the closed curve $\mathcal{C}$, with $\mathbf{n}$ and $\mathbf{T}$ oriented by the right hand rule)

$$\int\int_{\mathcal{S}} Curl(\mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds,$$

and Green's Theorem

$$\int\int_{\mathcal{R}_{xy}} N_x - M_y\, dA = \int_{\mathcal{C}} M dx + N dy.$$

(a) Derive Green's Theorem from Stokes Theorem.

Solution: Assume $\mathcal{S} = R_{xy}$ lies in the xy-plane, so its boundary $\mathcal{C}$ lies in the xy-plane as well, and assume $\mathbf{F} = \overrightarrow{(M, N, P)}$. Then $\mathbf{n} = \mathbf{k}$, and we have that $Curl\mathbf{F} = N_x - M_y$. This gives

$$\int\int_{\mathcal{S}} Curl(\mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int\int_{\mathcal{R}_{xy}} N_x - M_y\, dA,$$

which is equality on the left hand side. Using the differential form of the line integral we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\mathcal{C}} M dx + N dy + P dz\, ds = \int_{\mathcal{C}} M dx + N dy\, ds$$

because $dz = 0$ in the xy-plane. This gives equality of the right hand sides.

(b) Use Green's Theorem to construct a vector field $\mathbf{F}(x, y) = \overrightarrow{(M(x, y), N(x, y))}$ such that its line integral around any closed curve in the plane is equal to twice the area enclosed by the curve.

Solution: The left hand side of Green's Theorem gives twice the area of R_{xy} if $M_y - N_x = 2$. So choose any $\mathbf{F} = \overrightarrow{(M, N)}$ such that $M_y - N_x = 2$, for example, $\mathbf{F} = \overrightarrow{(y, -x)}$.

Problem #6 (25pts): Let $\mathcal{S}$ be the surface in $\mathcal{R}^3$ which is the image of $0 \leq u \leq 1$, $0 \leq v \leq 1$ under the parameterization $\mathbf{r}(u,v) = u^2\mathbf{i} - v^2\mathbf{j} + 2uv\mathbf{k}$.

(a) Find the unit normal $\mathbf{n}(u,v)$ that points toward the positive x-axis.

Solution: $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 2v \\ 0 & -2v & 2u \end{vmatrix} = \mathbf{i}(4v^2) - \mathbf{j}(4u^2) - \mathbf{k}(4uv) = 4\overrightarrow{(v^2, -u^2, -uv)}$$

Thus the normal that points toward positive x is $\mathbf{n} = \frac{\overrightarrow{(v^2, -u^2, -uv)}}{\sqrt{v^4+u^4+u^2v^2}}$

(b) Assume a fluid of constant density $\delta = \frac{1}{4}kg/m^3$ is moving toward positive x with velocity $\mathbf{F} = z\,\mathbf{i}$ in m/s. Determine the rate at which mass is flowing rightward through the surface $\mathcal{S}$. (Hint: Evaluate the flux integral

$$\int\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where $\mathbf{F} = \delta\mathbf{v}$ is the mass flux vector, and $\mathbf{n}$ points toward positive x.)

Solution:

$$\begin{aligned}
\int\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int_0^1 \int_0^1 \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \|\mathbf{r}_u \times \mathbf{r}_v\| dudv \\
&= \int_0^1 \int_0^1 \mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v\, dudv \\
&= \int_0^1 \int_0^1 \frac{1}{4} z \cdot 4\overrightarrow{(v^2, -u^2, -uv)}\, dudv \\
&= \int_0^1 \int_0^1 \frac{1}{4} z\overrightarrow{(1,0,0)} \cdot 4\overrightarrow{(v^2, -u^2, -uv)}\, dudv \\
&= \int_0^1 \int_0^1 zv^2\, dudv \\
&= 2\int_0^1 \int_0^1 uv^3\, dudv = \int_0^1 v^3\, dv = \frac{1}{4}\frac{kg}{s}
\end{aligned}$$

Problem #7 (25pts): Recall Newton's Gravitational Force Law for planetary motion:

$$\mathbf{F} = -\mathcal{G}\frac{M_S M_P}{r^3} = -\mathcal{G}\frac{\mathbf{r}}{r^3}$$

where M_S is the mass of the Sun, M_P is the mass of a planet, $\mathbf{r} = \overrightarrow{(x, y, z)}$, $r = \sqrt{x^2 + y^2 + z^2}$, and $\mathcal{G} = \mathcal{G} M_S M_P$. (The sun and planets are treated as point masses, and the origin of the coordinate system is at the Sun.)

(a) Show that $\nabla \frac{1}{r} = \frac{-\mathbf{r}}{r^3}$. (Hint: $\frac{\partial r}{\partial x} = \frac{x}{r}$, and similarly for y and z.) Use this to find a function f such that $\mathbf{F} = \nabla f$.

Solution:

$$\frac{\partial}{\partial x}\frac{1}{r} = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{x}{r^3}.$$

Similarly,

$$\frac{\partial}{\partial y}\frac{1}{r} = -\frac{y}{r^3}, \quad \frac{\partial}{\partial z}\frac{1}{r} = -\frac{z}{r^3}.$$

Thus

$$\nabla\left(\frac{G}{r}\right) = -G\frac{\overrightarrow{(x, y, z)}}{r^3}.$$

(b) A calculation would show that $Curl\mathbf{F} = 0$ for $r \neq 0$. Explain why $Curl\mathbf{F} = 0$ alone is enough to conclude that $\mathbf{F}$ is conservative even though it has a singularity at $r = 0$.

Solution: $r \neq$ is a simply connected region, and $Curl\mathbf{F} = 0$ in a simply connected region implies $\mathbf{F}$ is conservative.

Problem #8 (25pts): Recall Stokes Theorem:

$$\int\int_{\mathcal{S}} Curl(\mathbf{F})\cdot\mathbf{n}\,d\sigma = \int_{\mathcal{C}}\mathbf{F}\cdot\mathbf{T}\,ds, \tag{2}$$

where $\mathcal{S}$ is any 2-dimensional surface with closed curve $\mathcal{C}$ around its boundary. Recall also that if $\mathbf{F}=\mathbf{v}$ is the velocity field describing air streaming by, say, the wing of a plane, in steady state motion, and a bead were restricted to move along a frictionless closed wire hoop $\mathcal{C}$ fixed within the airflow, then

$$\int_{\mathcal{C}}\mathbf{v}\cdot\mathbf{T}\,ds = |\mathcal{C}|\ \bar{v},$$

where $\bar{v}$ is the average speed of the bead moving by the airflow around the hoop $\mathcal{C}$, and $|\mathcal{C}|$ is the length of $\mathcal{C}$.

Assume now that $\mathbf{F}=\mathbf{v}$ is in m/s, that $\mathbf{Curl}(\mathbf{v}) = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ is constant, and that $\mathcal{S}$ is the disk of radius $r=1$ with normal vector $\mathbf{i}+\mathbf{k}$. Use Stokes theorem together with the other assumptions to *derive* the frequency at which the bead would rotate around the hoop $\mathcal{C}$. (Hint: The frequency is $1/T$ where T is the time of one revolution.)

Solution: The time T it takes the bead to move one revolution around the hoop satisfies $\bar{v} = \frac{\text{Length of Hoop}}{T}$, so

$$\omega = \frac{1}{T} \ = \ \frac{\bar{v}}{\text{Length of Hoop}} = \frac{\bar{v}}{2\pi}$$

To find $\bar{v}$ use Stokes Theorem:

$$\begin{aligned}
|\mathcal{C}|\ \bar{v} &= 2\pi\bar{v} = \int_{\mathcal{C}}\mathbf{v}\cdot\mathbf{T}\,ds = \int\int_{\mathcal{S}} Curl(\mathbf{F})\cdot\mathbf{n}\,d\sigma \\
&= (Curl(\mathbf{F})\cdot\mathbf{n})\int\int_{\mathcal{S}} d\sigma = \pi Curl(\mathbf{F})\cdot\mathbf{n} \\
&= \pi\overrightarrow{(2,1,-1)}\cdot\frac{\overrightarrow{(1,0,1)}}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Conclude:

$$\bar{v} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4},$$

so

$$\omega = \frac{\bar{v}}{2\pi} = \frac{\sqrt{2}}{8\pi}\ \ \frac{revolutions}{second}.$$