Name: Solutions

Student ID#: ______________

Section: ______________

Final Exam

Monday June 14

MAT 21D, Temple, Spring 2012

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (25pts): Solve the following:
(a) Evaluate the integral $I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$. (Hint: Polar Coordinates)

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx \qquad I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta \underset{\substack{u=r^2\\ du=2r\,dr}}{=} \int_0^{2\pi} \frac{1}{2}\int_0^{\infty} e^{-u}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[-e^{-u}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{2\pi} (-0+1)\,d\theta$$

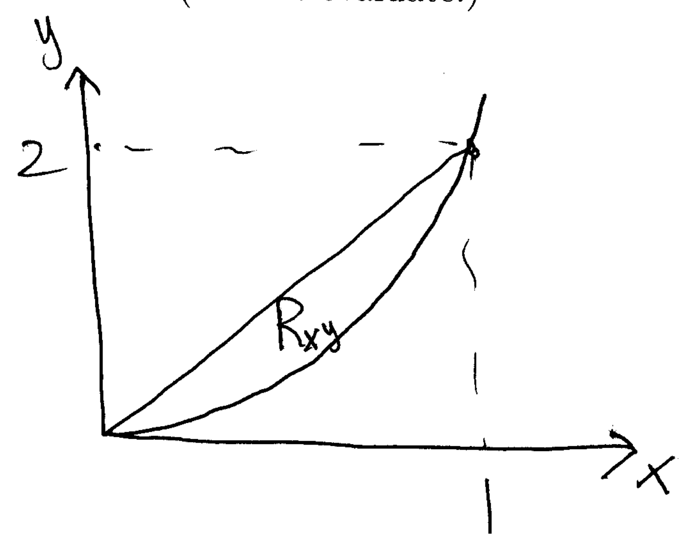
$$= \frac{1}{2}\cdot 2\pi = \pi \Rightarrow \boxed{I = \sqrt{\pi}}$$

#1 **(b)** Sketch the region of integration $\mathbf{R}_{xy}$ determined by the iterated integral

$$\int_0^1 \int_{2x^2}^{2x} \sqrt{x+y}(x-2y)^2 dy dx. \qquad (1)$$

and rewrite (1) with order of integration reversed. (Do not evaluate.)

$$\int_0^2 \int_{\frac{1}{2}y}^{\sqrt{y/2}} \sqrt{x+y}\,(x-2y)^2\,dx\,dy$$



$y = 2x \Leftrightarrow x = \frac{1}{2}y$

$y = 2x^2 \Leftrightarrow x = \sqrt{\frac{y}{2}}$

Problem #2 (25pts): A particle of mass $m = 2\ kg$ moves along a trajectory given by

$$\mathbf{r}(t) = 2\sin t\ \mathbf{i} + 2\cos t\ \mathbf{j} + t\ \mathbf{k},$$

where t is in seconds and $\mathbf{r}$ is in km. **(i)** At each time t find the following, give the correct dimensions, or say its dimensionless:

(a) The velocity vector $\mathbf{v}(t)$ $= 2\cos t\ \underset{\sim}{i} - 2\sin t\ \underset{\sim}{j} + \underset{\sim}{k}$

(b) The speed $v(t)$ $= \sqrt{4\cos^2 t + 4\sin^2 t + 1} = \sqrt{4+1} = \sqrt{5}$

(c) The acceleration vector $\mathbf{a}(t)$ $= -2\sin t\ \underset{\sim}{i} - 2\cos t\ \underset{\sim}{j}$

(d) The unit tangent vector $\mathbf{T}(t)$ $= \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{2}{\sqrt{5}}\cos t\ \underset{\sim}{i} - \dfrac{2}{\sqrt{5}}\sin t\ \underset{\sim}{j} + \dfrac{1}{\sqrt{5}}\underset{\sim}{k}$

(e) The unit normal $\mathbf{N}(t)$ $= \dfrac{\dot{T}}{|\dot{T}|} = \dfrac{\frac{1}{\sqrt{5}}(-2\sin t, -2\cos t, 0)}{\frac{1}{\sqrt{5}}\sqrt{2^2\sin^2 t + 2^2\cos^2 t}}$

$= -\sin t\ \underset{\sim}{i} - \cos t\ \underset{\sim}{j}$

(f) The curvature $\kappa(t)$.

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{1}{|v|}\left|\frac{dT}{dt}\right| = \frac{1}{\sqrt{5}} \cdot \left|\frac{1}{\sqrt{5}} \cdot 2\right| = \frac{2}{5}$$

(g) The length of the component of $\mathbf{a}(t)$ in direction of $\mathbf{T}$.

$$\vec{a} \cdot \vec{T} = (-2\sin t, -2\cos t) \cdot \left(\tfrac{2}{5}\cos t, -\tfrac{2}{5}\sin t, \tfrac{1}{\sqrt{5}}\right) = 0$$

#2 **(ii)** Find the arclength from $t = 0$ to $t = 1$.

$$ds = |\vec{v}|\,dt = \sqrt{5}\,dt$$

$$\int_0^1 |\vec{v}|\,dt = \int_0^1 \sqrt{5}\,dt = \sqrt{5}$$

#2 **(iii)** Write down and evaluate the correct line integral for the work done by the Force creating the motion between $t = 0$ to $t = 1$. Include the correct units.

$$\int_{t=0}^{t=1} \vec{F}\cdot\vec{T}\,ds = \int_0^1 \vec{F}\cdot\vec{v}\,dt =$$

$$\vec{F} = m\vec{a} = \overrightarrow{(-4\sin t, -4\cos t, 0)}$$

$$= \int_0^1 \overrightarrow{(-4\sin t, -4\cos t, 0)} \cdot \overrightarrow{(2\cos t, -2\sin t, 1)}\,dt = 0$$

$$= \int_0^1 -8\cos t\sin t - 8\sin t\cos t\,dt = -16\int_0^1 \sin t\cos t\,dt$$

$$u = \sin t \quad du = \cos t\,dt$$

$$= -16\int_{t=0}^{t=1} u\,du = -8u^2\Big]_{t=0}^{t=1} = -8\sin^2 t\Big]_0^1$$

$$= -8\sin^2(1)\ \frac{kg\,m^2}{s^2}$$

Problem #3 (25pts): (a) Use the Leibniz substitution principle to show that in general, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a parameterization of curve C for $a \leq t \leq b$, and $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field, then the following are equal:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_C M dx + N dy + P dz.$$

$$\int_C \vec{F} \cdot \vec{T}\, ds \overset{\uparrow}{=} \int_C \vec{F} \cdot \vec{T} |\vec{v}|\, dt = \int_C \vec{F} \cdot \underbrace{\vec{v}\, dt}_{d\vec{r}} = \int_C \vec{F} \cdot d\vec{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{F} \cdot \vec{v} = M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}$$

$$\frac{ds}{dt} = |\vec{v}| \qquad \vec{F} \cdot \vec{v}\, dt = M dx + N dy + P dz$$

$$\therefore \int_C \vec{F} \cdot \vec{v}\, dt = \int_C M dx + N dy + P dz$$

#3 **(b)** Assume further that $F = \nabla f$ for some scalar function $f(x, y, z)$. Prove that

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

$$\int_{C} \vec{F} \cdot \vec{T}\, ds = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} \nabla f(\vec{r}(t))\, \dot{\vec{r}}(t)\, dt$$

$$= \int_{t_1}^{t_2} \frac{d}{dt} f(\vec{r}(t))\, dt = f(\vec{r}(t_2)) - f(\vec{r}(t_1))$$

4

Problem #~~5~~ (25pts): Let

$$\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2 + z)\mathbf{j} + (y + \cos z)\mathbf{k}.$$

Find:

(a) Div **F** $= 2y + 0 - \sin z = 2y - \sin z$

(b) Curl **F** $= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2+z & y+\cos z \end{vmatrix} =$

$= \underset{\sim}{i}(0) - \underset{\sim}{j}(0) + \underset{\sim}{k}(2x - 2x) = 0$

#4 **(c)** f such that $\mathbf{F} = \nabla f$

$$f = \int_x 2xy \, dx = x^2 y + g(y,z)$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 + z \Rightarrow \frac{\partial g}{\partial y} = z \Rightarrow g = zy + h(z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2 y + zy + h(z)\right) = y + h'(z) = y + \cos z$$

$$h'(z) = \cos z \Rightarrow h(z) = \sin z$$

$$\Rightarrow f(x,y,z) = x^2 y + zy + \sin z$$

#4 **(d)** Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where C is the *hypo-gastro-geometric-meso-cycloid*, a smooth curve that takes $A = (1, -1, \underset{\pi}{\cancel{2}})$ to $B = (-1, 1, \underset{0}{\cancel{1}})$.

$$\int_C F\cdot T\,ds = f(-1,1,0) - f(1,-1,\pi)$$

$$= (-1)^2\cdot 1 + 0 + 0 - \left[(1)^2\cdot(-1) + \pi(-1) + \sin\pi\right]$$

$$= \cancel{1} + 1 + \pi + \cancel{\sin\pi} = \boxed{1+\pi}$$

Problem #~~4~~ 5 (25pts): Let $\mathcal{S}$ be the surface in $\mathcal{R}^3$ which is the image of $0 \leq u \leq 1$, $0 \leq v \leq 2$ under the parameterization

$$\mathbf{r}(u,v) = u^2\mathbf{i} + uv\mathbf{j} + v^2\mathbf{k}.$$

$$r_u = (2u, v, 0)$$

$$r_v = (0, u, 2v)$$

Evaluate the flux integral

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, d\sigma,$$

where $\mathbf{F}$ is the vector field

$$\mathbf{F} = \sqrt{x}\,\mathbf{i}.$$

$$\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\, d\sigma = \int_0^2\int_0^1 \vec{F}\cdot\frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cdot \cancel{|\vec{r}_u \times \vec{r}_v|}\, du\, dv$$

$$= \int_0^2\int_0^1 \overrightarrow{(u,0,0)} \cdot \overrightarrow{(2v^2, -4uv, 2u^2)}\, du\, dv$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2u & v & 0 \\ 0 & u & 2v \end{vmatrix} = \underset{\sim}{i}(2v^2) - \underset{\sim}{j}(4uv) + \underset{\sim}{k}(2u^2)$$

$$= \int_0^2\int_0^1 2uv^2\, du\, dv = \int_0^2\int_0^1 u^2 v\Big]_{u=0}^{u=1} du\, dv = \int_0^2 v\, dv$$

$$= \frac{1}{2}v^2\Big]_0^2 = \boxed{2}$$

6

Problem #~~7~~ (25pts): Let $\mathbf{x} = (x, y, z)$, assume that a density $\rho(\mathbf{x}, t)$ evolves according to the velocity field $\mathbf{v}(\mathbf{x}, t)$, and recall that the mass flux vector is the vector $\mathbf{F} = \rho\mathbf{v}$.

(a) Derive the dimensional units of $\mathbf{F}$, define the flux of $\mathbf{F} = \rho\mathbf{v}$ through a surface $\mathcal{S}$, and describe the physical meaning of this flux.

$$[\rho v] = \frac{M}{L^2 T} = \frac{\text{mass}}{\text{area time}}$$

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, d\sigma = \sum \frac{\text{mass}}{\cancel{\text{area}}\ \text{time}} \times \cancel{\text{area}}$$

= mass per time thru $\mathcal{S}$ in direction $\vec{n}$

#6 **(b)** Assume $\rho(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$ satisfy the continuity equation

$$\rho_t + Div(\rho\mathbf{v}) = 0.$$

Use the Divergence Theorem to show that for any given volume $\mathcal{V}$, the principle of *conservation of mass* holds in $\mathcal{V}$. (In words, the principle states that the time rate of change of total mass inside a volume $\mathcal{V}$ is equal to minus the flux of mass out through the boundary $\partial\mathcal{V}$.)

$$0 = \iiint_{\mathcal{V}} \rho_t + Div(\rho\vec{v})\, dV = \iiint_{\mathcal{V}} \rho_t \, dV + \iiint_{\mathcal{V}} Div\, \rho\vec{v}\, dV$$

$$= \underbrace{\frac{d}{dt}\iiint_{\mathcal{V}} \rho\, dV}_{\text{time rate of chng of mass in } \mathcal{V}} + \underbrace{\iint_{\partial\mathcal{V}} \rho\vec{v}\cdot\vec{n}\, ds}_{\text{rate at which mass passing out thru } \partial\mathcal{V}} = 0$$

7

Problem #~~8~~ (25pts): Part I: In words, the value of the $Curl\mathbf{F} \cdot \mathbf{n}$ of a vector valued function $\mathbf{F}$ dotted with unit normal $\mathbf{n}$ at a point $\mathbf{x_0} = (x_0, y_0, z_0)$ is "the circulation in $\mathbf{F}$ per area around axis $\mathbf{n}$" at $\mathbf{x_0}$. Given this, show that the maximal circulation per area occurs around an axis parallel to the $Curl\mathbf{F}$, and that the length of $Curl\mathbf{F}$ is that maximal circulation per area.

$$\text{Curl}\vec{F} \cdot \vec{n} = \|\text{Curl}\vec{F}\| \|\vec{n}\| \cos\theta$$

$$= \|\text{Curl}\vec{F}\| \cos\theta$$

where $\theta = \angle$ betw $\text{Curl}\vec{F}$ & $\vec{n}$.

Thus max'al when $\theta = 0$ & $\vec{n} = \dfrac{\text{Curl}\vec{F}}{\|\text{Curl}\vec{F}\|}$

$\therefore$ Max'al Circ. per area is

$$\text{Curl}\vec{F} \cdot \frac{\text{Curl}\vec{F}}{\|\text{Curl}\vec{F}\|} = \frac{\|\text{Curl}\vec{F}\|^2}{\|\text{Curl}\vec{F}\|} = \|\text{Curl}\vec{F}\|$$

#7 Part II: Let $\mathbf{F} = x^2\mathbf{i} - xy\mathbf{j} + z\mathbf{k}$. Find the circulation per area around the the axis $\mathbf{i} + \mathbf{k}$ at the point $P_0 = (1, -2, 3)$.

$$\text{Curl}F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & -xy & z \end{vmatrix} = \underset{\sim}{i}(0) - \underset{\sim}{j}(0) + \underset{\sim}{k}(-y)$$

$$= -y\underset{\sim}{k}$$

$$\text{Curl}\vec{F}\cdot\vec{n} = -y\underset{\sim}{k} \cdot \frac{\underset{\sim}{i} + \underset{\sim}{k}}{\sqrt{2}} = -\frac{y}{\sqrt{2}} = \boxed{-\frac{3}{\sqrt{2}}}$$

8

Problem #$\cancel{6}$ (25pts): Verify Stokes Theorem:

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}ds = \int\int_{\mathcal{S}} Curl\mathbf{F}\cdot\mathbf{n}\,d\sigma,$$

where $\mathcal{S}$ is the hemisphere $x^2+y^2+z^2=9,\quad z\ge 0,\quad \mathbf{F}=y\mathbf{i}-x\mathbf{j}$, and $\mathcal{C}$ is the closed curve $x^2+y^2=3$.

$$\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \vec{F}\cdot\vec{v}\,dt = \int_0^{2\pi}(3\sin t,-3\cos t)\cdot(-3\sin t,3\cos t)dt$$

$$\vec{r}(t) = 3(\cos t\,\underset{\sim}{i} + \sin t\,\underset{\sim}{j}),\quad 0\le t\le 2\pi$$

$$\vec{v}(t) = 3(-\sin t\,\underset{\sim}{i} + \cos t\,\underset{\sim}{j})$$

$$= \int_0^{2\pi} -9\sin^2 t - 9\cos^2 t\,dt = -9\cdot 2\pi = -18\pi$$

$$\iint_{\mathcal{S}} Curl\vec{F}\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi/2} Curl\vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{\cancel{|\vec{r}_u\times\vec{r}_v|}}\,|\cancel{\vec{r}_u\times\vec{r}_v}|\,dv\,du$$

$$\vec{r}(\theta,\varphi) = 3\cos\theta\sin\varphi\,\underset{\sim}{i} + 3\sin\theta\sin\varphi\,\underset{\sim}{j} + 3\cos\varphi\,\underset{\sim}{k}$$

$$u=\theta,\ v=\varphi\quad |\vec{r}_\theta\times\vec{r}_u| = 9\sin\varphi$$

$$\cancel{\vec{r}_\theta = -3\sin\theta\sin\varphi\,\underset{\sim}{i} + 3\cos\theta\sin\varphi\,\underset{\sim}{j}}$$

$$\cancel{\vec{r}_\varphi = 3\cos\theta\cos\varphi\,\underset{\sim}{i} + 3\sin\theta\cos\varphi\,\underset{\sim}{j} - 3\sin\varphi\,\underset{\sim}{k}}$$

16 $\cancel{0\le\theta\le 2\pi,\ 0\le\varphi\le\frac{\pi}{2}}$

(#8 cont)

Part II: Let $\mathbf{F} = x^2\mathbf{i} - xy\mathbf{j} + z\mathbf{k}$. Find the circulation per area around the the axis $\mathbf{i}+\mathbf{k}$ at the point $P_0 = (1,-2,3)$.

$$\operatorname{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = -2\underset{\sim}{k}$$

$$\vec{n} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{3}\overrightarrow{(x,y,z)}$$

$$\operatorname{Curl}\vec{F}\cdot\vec{n} = \frac{1}{3}\overrightarrow{(x,y,z)}\cdot\overrightarrow{(0,0,-2)} = -\frac{2}{3}z$$

$$\iint_S \operatorname{Curl}\vec{F}\cdot\vec{n}\,d\sigma = \int_0^{\pi/2}\int_0^{2\pi} \underbrace{-\frac{2}{3}\cdot 3\cos\varphi}_{-\frac{2}{3}z}\cdot 9\sin\varphi\,d\theta\,d\varphi$$

$$= -18\int_0^{\pi/2} 2\pi\cos\varphi\sin\varphi\,d\varphi \underset{\uparrow}{=} +18\cdot 2\pi\,\frac{\cos^2\varphi}{2}\Big]_0^{\pi/2} = -18\pi \checkmark$$

$u = \cos\varphi$

$du = -\sin\varphi\,d\varphi$