Name: ______________________

Student ID#: ______________________

Section: ______________________

Final Exam

Monday March 19, 3:30-5:30pm

MAT 21D, Temple, Winter 2018

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (25pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ and evaluate the iterated integral

$$\int_{-1}^{0}\int_{x^2}^{-x} x^2 y\, dydx. \tag{1}$$

Solution:

$$\begin{aligned}
\int_{-1}^{0}\int_{x^2}^{-x} x^2 y\, dydx &= \int_{-1}^{0} x^2 \left(\frac{y^2}{2}\right)\Bigg|_{y=x^2}^{y=-x} dx = \frac{1}{2}\int_{-1}^{0}\left(x^4 - x^6\right)\, dx \\
&= \frac{1}{2}\left(\frac{x^5}{5} - \frac{x^7}{7}\right)\Bigg|_{x=-1}^{x=0} = -\frac{1}{2}\left(\frac{(-1)^5}{5} - \frac{(-1)^7}{7}\right) \\
&= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{35}.
\end{aligned}$$

(b) Rewrite (1) with order of integration reversed. (Do not re-evaluate).

Solution:

$$\int_{-1}^{0}\int_{x^2}^{-x} x^2 y\, dydx = \int_{-\sqrt{y}}^{-y}\int_{-1}^{0} x^2 y\, dxdy.$$

Problem #2 (25pts): Use spherical coordinates (ρ, ϕ, θ) to find the volume of the region obtained by removing the cone $\phi \leq \pi/4$ from the sphere $x^2 + y^2 + z^2 = 9$.

Solution:

$$\begin{aligned}
\int_{\pi/4}^{\pi} \int_{0}^{2\pi} \int_{0}^{3} \rho^2 \sin\phi \, d\phi d\theta d\rho &= 2\pi \int_{\pi/4}^{\pi} \sin\phi \, d\phi \int_0^3 \rho^2 \, d\rho = 2\pi \left(-\cos\phi\right)_{\pi/4}^{\pi} \left(\frac{\rho^3}{3}\right)_0^3 \\
&= 2\pi \left(1 + \frac{\sqrt{2}}{2}\right)\left(\frac{27}{3}\right)
\end{aligned}$$

Problem #3 (25pts): A cannonball is shot out of a canon at angle α to the ground. Assuming the acceleration of gravity is exactly $g = 10m/s^2$, and the muzzle velocity of the cannonball is $100m/s$, find the angle α such that the cannonball will hit the ground at a distance of exactly 500 meters.

Solution: The equations are $\ddot{y}(t) = -g$, $\ddot{x} = 0$, so integrating twice, and assuming the initial position is $x_0 = 0$, $y_0 = 0$ at $t = 0$, gives

$$y(t) = -\frac{1}{2}gt^2 + v_y t, \qquad x(t) = v_x t,$$

where $v_x = v\cos\alpha$ and $v_y = v\sin\alpha$, with $v = 100$ and $g = 10$. Now the time t_* at which $x = 500$ solves $500 = v_x t_*$, so

$$t_* = \frac{500}{v_x}.$$

Asking that $y = 0$ at $t = t_*$ gives $0 = -\frac{1}{2}gt_*^2 + v_y t_*$, so dividing by t_* gives

$$0 = -\frac{1}{2}g\left(\frac{500}{v_x}\right) + v_y.$$

multiplying by $2v_x$ and moving $v_x v_y$ to the other side gives

$$2v_x v_y = v^2 2\cos\alpha\sin\alpha = 500g.$$

Thus by the double angle formula, using $v^2 = 10,000$, $g = 10$, we obtain

$$\sin 2\alpha = \frac{500g}{v^2} = \frac{5000}{10,000} = \frac{1}{2}.$$

Conclude: $2\alpha = 30^o$, and so $\alpha = 15^o = \pi/12$.

Problem #4 (20pts): (a) Let $F = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$, let C be a smooth curve that takes A to B, and let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Use Leibniz's substitution principle to show the following are equal:

$$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds = \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{v}}\,dt = \int_C M dx + N dy + P dz.$$

Solution: Since $\vec{\mathbf{v}} = \frac{ds}{dt}\vec{\mathbf{T}}$, formally substituting $\vec{\mathbf{v}}dt$ for $\vec{\mathbf{T}}ds$ into the first integral gives the second. Since $\vec{\mathbf{v}} = d\vec{\mathbf{r}}/dt$, formally substituting $d\vec{\mathbf{r}}$ for $\vec{\mathbf{v}}dt$ into the second integral gives the third. Finally, since

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

substituting $\vec{\mathbf{F}} = \overrightarrow{(M, N, P)}$ and $d\vec{\mathbf{r}} = \overrightarrow{(dx, dy, dz)}$ into the third integral gives the fourth.

(b) Assume further that $\mathbf{F} = m\mathbf{a}$, and $\vec{\mathbf{F}}$ is conservative, so $\vec{\mathbf{F}} = -\nabla P$. Derive the principle of conservation of energy

$$\left\{\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right\} + \{P(B) - P(A)\} = 0. \tag{2}$$

(Hint: Integrate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ two different ways.)

Solution: Given $\mathbf{r}(t)$ with $m\mathbf{r}''(t) = \mathbf{F}$, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\; ds &= \int_C m\mathbf{v}'(t) \cdot \mathbf{v}(t)\; dt = \int_C m\frac{1}{2}\frac{d}{dt}\left(\mathbf{v}(t) \cdot \mathbf{v}(t)\right)\; dt \\ &= \frac{1}{2}mv^2|_{t_A}^{t_B} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_B^2.\end{aligned}$$

Integrating another way,

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_{t_A}^{t_B} -\nabla P \cdot \mathbf{v}\, dt = -\int_{t_A}^{t_B} \frac{d}{dt}P(\mathbf{r}(t)\, dt = -\left(P(B) - P(A)\right).$$

Setting these two expressions for $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, and putting everything on the left hand side of the equation, gives (2).

Problem #5 (25pts): Recall Stokes Theorem:

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_{\mathcal{S}} Curl(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

and Green's Theorem

$$\int_{\mathcal{C}} M dx + N dy = \int\int_{\mathcal{R}_{xy}} N_x - M_y\, dA.$$

(a) Use the definition of the *Curl* to derive Green's Theorem from Stokes Theorem.

Solution:

$$Curl\vec{\mathbf{F}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial y & \partial z \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\mathbf{k}, \tag{3}$$

so putting $Curl\vec{\mathbf{F}} \cdot \mathbf{n} = (N_x - M_y)$ into the right hand side of Stokes Theorem, and assuming the surface S lies in the (x, y) plane, (so $S = R_{xy}$), gives the RHS of Green's theorem.

For the left hand side, use the fourth equivalence in part (**a**).

(b) Let a be any given real number. Use Green's Theorem to construct a vector field $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ such that

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = aA,$$

where $A > 0$ is the area enclosed by the simple closed curve C.

Solution: We need M and N so that $N_x - M_y = a$. So choose $N = \frac{ax}{2}$, $M = \frac{ay}{2}$.

(c) What are the possible values for a if $\vec{\mathbf{F}}$ is a *conservative* vector field? Explain.

Solution: If $\vec{\mathbf{F}}$ is conservative, then $Curl\vec{\mathbf{F}} = 0$, so the right hand side of Stokes Theorem is zero. Therefore the only possible value is $a = 0$.

Problem #6 (20pts): (a) Use Stokes Theorem to derive the meaning of $Curl\,\vec{\mathbf{F}}(\mathbf{x}_0)\cdot\mathbf{n}$ at a point $\mathbf{x}_0 = (x_0, y_0, z_0)$ as the circulation per area around axis $\mathbf{n}$ at $\mathbf{x}_0$.

Solution: Let $D_\epsilon \equiv D_\epsilon(\mathbf{x}_0)$ be the disc of radius ϵ, normal $\mathbf{n}$, and center $\mathbf{x}_0$, and let C_ϵ denote the circular boundary of D_ϵ. Applying Stokes Theorem gives:

$$\int_{C_\epsilon} \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds = \int\int_{D_\epsilon} Curl\vec{\mathbf{F}}\cdot\mathbf{n}\,d\sigma. \tag{4}$$

Now for ϵ small, the $Curl\vec{\mathbf{F}}\cdot\mathbf{n}$ on the RHS tends to $Curl\vec{\mathbf{F}}(\mathbf{x}_0)\cdot\mathbf{n}$. Neglecting the errors which will tend to zero as $\epsilon \to 0$, we can pull the constant out of the RHS, and solve for $Curl\vec{\mathbf{F}}(\mathbf{x}_0)\cdot\mathbf{n}$, to obtain

$$Curl\vec{\mathbf{F}}(\mathbf{x}_0)\cdot\mathbf{n} = \frac{\int_{C_\epsilon}\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}ds}{Area\,(D_\epsilon)} + error,$$

where the error tends to zero as $\epsilon \to 0$. Conclude:

$$Curl\vec{\mathbf{F}}(\mathbf{x}_0)\cdot\mathbf{n} = \lim_{\epsilon\to 0}\frac{\int_{C_\epsilon}\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}ds}{Area\,(D_\epsilon)}, \tag{5}$$

which shows that $Curl\vec{\mathbf{F}}(\mathbf{x}_0)\cdot\mathbf{n}$ is the circulation per area around axis $\mathbf{n}$ at $\mathbf{x}_0$.

(b) Use the Divergence Theorem to derive the meaning of the $Div\vec{\mathbf{F}}(\mathbf{x}_0)$ at a point $\mathbf{x}_0 = (x_0, y_0, z_0)$ as the Flux per volume at $\mathbf{x}_0$.

Solution: Let $B_\epsilon \equiv B_\epsilon(\mathbf{x}_0)$ be the ball of radius ϵ center $\mathbf{x}_0$, and let S_ϵ denote the spherical boundary of B_ϵ. Applying Divergence Theorem gives:

$$\int\int_{S_\epsilon} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma = \int\int\int_{B_\epsilon} Div\vec{\mathbf{F}}\, dV. \tag{6}$$

Now for ϵ small, the $Div\vec{\mathbf{F}}$ on the RHS tends to $Div\vec{\mathbf{F}}(\mathbf{x}_0)$. Neglecting the errors which will tend to zero as $\epsilon \to 0$, we can pull the constant out of the RHS, and solve for $Div\vec{\mathbf{F}}(\mathbf{x}_0)$, to obtain

$$Div\vec{\mathbf{F}}(\mathbf{x}_0) = \frac{\int\int_{S_\epsilon} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} d\sigma}{Vol\,(B_\epsilon)} + error,$$

where the error tends to zero as $\epsilon \to 0$. Conclude:

$$Div\vec{\mathbf{F}}(\mathbf{x}_0) = \lim_{\epsilon \to 0} \frac{\int\int_{S_\epsilon} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} d\sigma}{Vol\,(B_\epsilon)}, \tag{7}$$

which shows that $Div\vec{\mathbf{F}}(\mathbf{x}_0)$ is the flux per volume at $\mathbf{x}_0$.

Problem #7 (25pts): Let $\delta \equiv \delta(x, y, z, t)$ denote the density of a gas in motion, (you can think of air flowing around an airplane), and assume it is being transported by velocity

$$\mathbf{v} = \mathbf{v}(x, y, z, t) = M(x, y, z, t)\mathbf{i} + N(x, y, z, t)\mathbf{j} + P(x, y, z, t)\mathbf{k}.$$

Let $\vec{\mathbf{F}} = \delta\mathbf{v}$ denote the mass flux vector. Now not every density and velocity field can be a real flow, and the condition it must satisfy in order that mass be conserved, is the continuity equation

$$\delta_t + Div(\delta\mathbf{v}) = 0. \tag{8}$$

Use the Divergence Theorem, and the physical interpretation of Flux, to show that if (8) holds, then mass is conserved in every volume $V \subset \mathbf{R}^2$. (Hint, show the rate at which mass changes in V equals the rate at which mass is flowing out through the boundary, at each fixed time. Use enough English words to give an argument that makes sense.)

Solution: Fix an arbitrary volume $V \subset \mathbf{R}^3$. We show that (8) implies that mass is conserved in V. For this, integrate (8) over V to get

$$\begin{aligned} 0 &= \int\int\int_V \delta_t + Div(\delta\vec{\mathbf{v}})\, dV \\ &= \int\int\int_V \delta_t\, dV + \int\int\int_V Div(\delta\vec{\mathbf{v}})\, dV \\ &= \frac{d}{dt}\int\int\int_V \delta\, dV + \int\int\int_V Div(\delta\vec{\mathbf{v}})\, dV \\ &= \frac{d}{dt}\int\int\int_V \delta\, dV + \int\int_{\partial V} \delta\vec{\mathbf{v}}\cdot\mathbf{n}\, d\sigma \end{aligned}$$

so

$$\frac{d}{dt}\int\int\int_V \delta\, dV = -\int\int_{\partial V} \delta\vec{\mathbf{v}}\cdot\mathbf{n}\, d\sigma.$$

This last equation expresses that the time rate of change of the mass in V is equal to the rate at which mass is flowing outward through the boundary, meaning that mass is conserved in V. Since (8) implies mass is conserved in every volume, we conclude that (8) implies conservation of mass.

Problem #8 (25pts): Let S denote the two dimensional surface in the (x, y)-plane bounded by the ellipse $a^2\, x^2 + b^2\, y^2 = 1$. Verify Stokes Theorem $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \int\int_S Curl \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} d\sigma$ for $\vec{\mathbf{F}} = y\mathbf{i} - x\mathbf{j}$ by directly evaluating both sides by explicit parametrization, and showing they are equal.

(a) $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds =?$

Solution: For a parameterization, use $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$. Then

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \int_0^{2\pi} \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} dt = \int_0^{2\pi} (b\sin t, -a\cos t) \cdot (-a\sin t, b\cos t) dt = -2\pi ab.$$

(b) $\int\int_S Curl\vec{\mathbf{F}} \cdot \vec{\mathbf{n}} d\sigma =$

Solution: Let $u = ax$, $v = by$. Then $d\sigma = dxdy = \frac{\partial(u,v)}{\partial(x,y)} dudv = ab\, dudv$, $\vec{\mathbf{n}} = \mathbf{k}$, and $Curl\vec{\mathbf{F}} = (N_x - M_y)\mathbf{k} = -2\mathbf{k}$. Therefore,

$$\int\int_S Curl\vec{\mathbf{F}} \cdot \vec{\mathbf{n}} d\sigma = \int\int_{u^2+v^2\leq 1} -2\mathbf{k} \cdot \mathbf{k}\, ab\, dudv = -2\pi ab.$$