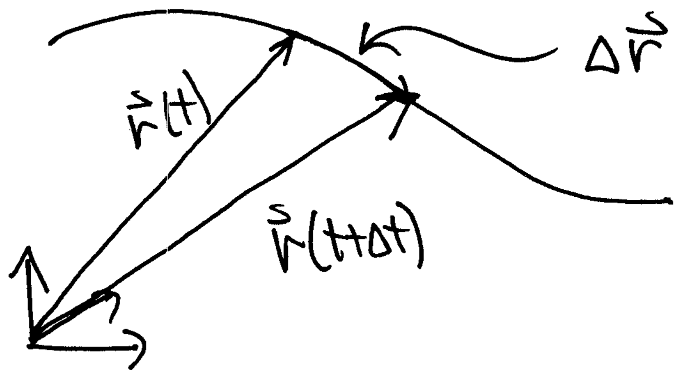
Arclength / Unit tangent & normal

13.4

- Vector valued function -

$$\vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} + z(t)\,\underset{\sim}{k}$$

$$\dot{\vec{r}}(t) = \vec{r}\,'(t) = \lim_{\Delta t \to 0}\left[\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{\Delta\vec{r}}{\Delta t}\right]$$



- geometrically - $\Delta\vec{r}$ points along the curve

<u>calculationally</u> - $\vec{r}\,' = x'(t)\,\underset{\sim}{i} + y'(t)\,\underset{\sim}{j} + z'(t)\,\underset{\sim}{k}$

From this we get –

1. $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$ points tanget to curve

2. $\|\vec{v}\| = \lim_{\Delta t \to 0} \frac{\|\Delta \vec{r}\|}{\Delta t} = \frac{ds}{dt}$ = speed

The acceleration $\vec{a}$ is more complicated & can point in any direction / any length.

- There are two basic directions along a curve: $\vec{T}$ = unit tangent vector (gives direction of motion)

  $\vec{N}$ = unit normal (gives direction curve is "curving")

- Unit Tangent vector –

Recall – to find a unit vector in direction $\vec{v}$, divide by its length –

$$\vec{v} = \overrightarrow{(a,b,c)} \qquad \|\vec{v}\| = \sqrt{a^2+b^2+c^2}$$

$$\frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

$$\left\|\frac{\vec{v}}{\|\vec{v}\|}\right\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1 \checkmark$$

Defn: $\boxed{\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}}$

Ex: $\vec{r}(t) = 2\cos t \, \underset{\sim}{i} + 2\sin(t) \, \underset{\sim}{j} + t \, \underset{\sim}{k}$

$$\vec{v}(t) = -2\sin t \, \underset{\sim}{i} + 2\cos t \, \underset{\sim}{j} + \underset{\sim}{k}$$

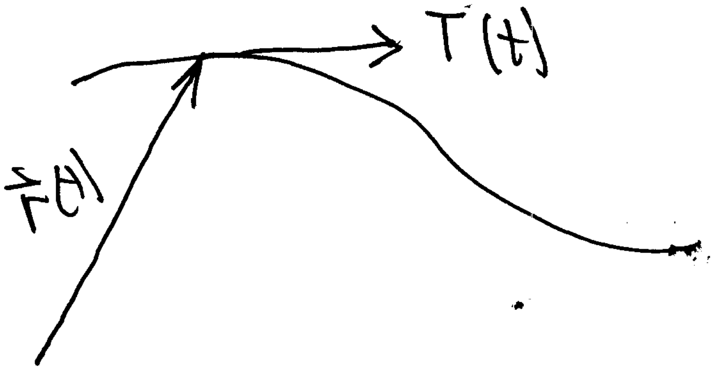
$$T(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} \qquad \|\vec{v}\| = v = \sqrt{4\sin^2 t + 4\cos^2 t + 1}$$

$$= \sqrt{5}$$

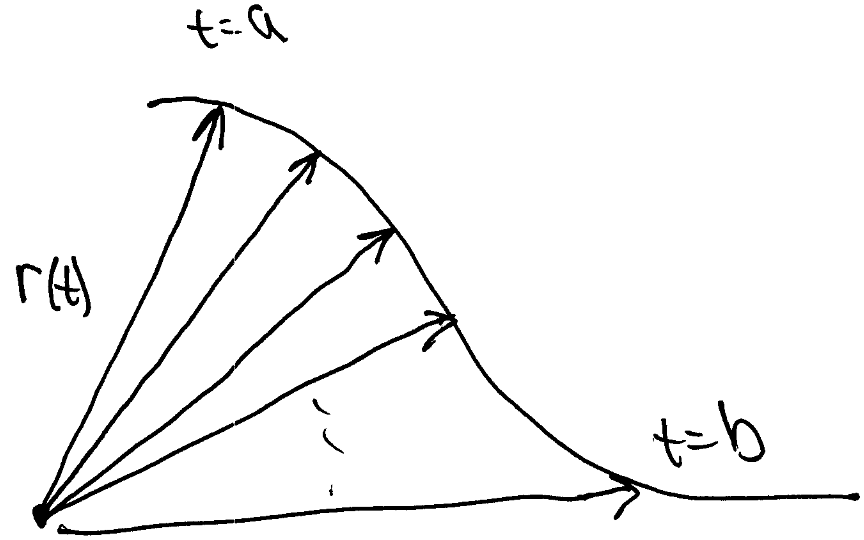
Check: $T(t) = \dfrac{-2\sin t}{\sqrt{5}} \underset{\sim}{i} + \dfrac{2\cos t}{\sqrt{5}} \underset{\sim}{j} + \dfrac{1}{\sqrt{5}} \underset{\sim}{k}$

$$\|T(t)\|^2 = \frac{4\sin^2 t + 4\cos^2 t + 1}{(\sqrt{5})^2} = 1 \checkmark$$

- $T(t)$ gives the direction of motion along curve



- Arclength:



Break curve into segments according to

$$t_0 = a \le t_1 \le t_2 \le \cdots \le t_N = b \qquad t_k = a + k\Delta t$$

$$\|\vec{r}(t_k) - \vec{r}(t_{k-1})\| \approx \text{length of segment}$$

$$\text{Length}_a^b \approx \sum_{k=1}^{N} \underbrace{\frac{\|\vec{r}(t_k) - \vec{r}(t_{k-1})\|}{\Delta t}}_{\|\vec{v}(t)\|} \Delta t$$

Riemann Sum for

$$\boxed{L_a^b = \int_a^b \|\vec{v}(t)\|\, dt}$$

That is - integrate the speed

$$\|\vec{v}(t)\| = \frac{ds}{dt}$$

to get the arclength

$$\int_a^b \frac{ds}{dt}\,dt = \int_a^b ds$$

$$\|\vec{v}\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{\sqrt{dx^2+dy^2+dz^2}}{dt}$$

$$ds = \sqrt{dx^2+dy^2+dz^2}$$

all say same thing...

Ex: Find length of helix

$$\vec{r}(t) = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j} + t\, \underset{\sim}{k}$$

for $0 \le t \le 2\pi$.

Soln: $ds = \sqrt{dx^2 + dy^2 + dz^2}$

$x = \cos t \quad dx = -\sin t\, dt$

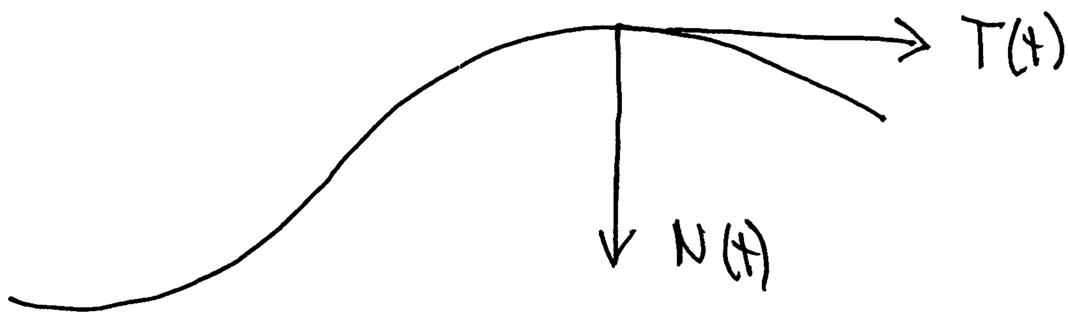
$y = \sin t \quad dy = \cos t\, dt$

$z = t \quad dz = dt$

$$ds = \sqrt{\cos^2 t + \sin^2 t + 1}\; dt$$

$$L_0^{2\pi} = \int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{2}\, dt = \boxed{\sqrt{2} \cdot 2\pi}$$

- Where does the acceleration point?



$N(t)$ = principle normal gives direction in which the "curve is curving"

To find it: recall:

<u>Thm</u>: if $\|T(t)\| = \text{const}$, then $\dot{T}(t) \perp T(t)$

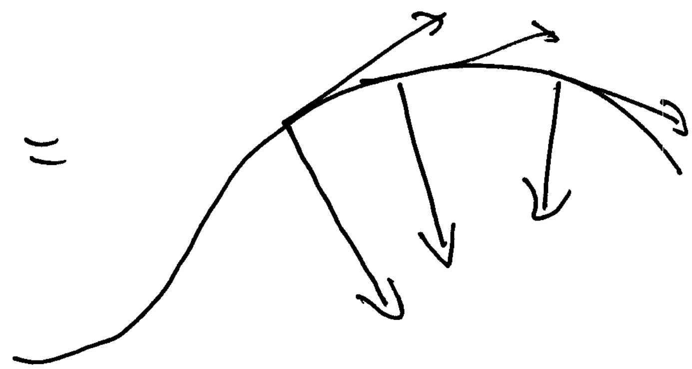
Pf. $T(t) \cdot T(t) = \text{const}$

$$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0$$

$$2T' \cdot T = 0 \qquad T' \perp T$$

- <u>Conclude</u>: The vector $T'(t) = \frac{dT}{dt}$ points $\perp$ to $T(t)$ and gives the "direction of curvature"

- <u>Defn</u>: $N(t) = \frac{T'(t)}{\|T'(t)\|} =$ unit vector in direction of $T'(t)$ is direction of curvature



$N(t) =$ "principal normal vector"

- <u>Defn</u>: $\|T'(t)\| = \kappa v =$ curvature.

(measures magnitude of curving)

Ie: $\left\|\frac{dT}{ds}\right\| = \kappa \quad \left\|\frac{dT}{dt}\right\| = \left\|\frac{ds}{dt}\frac{dT}{ds}\right\| = v\kappa \cancel{N}$

<u>Theorem</u>: $\vec{a}(t) = \frac{d^2s}{dt^2} T(t) + v^2 K N(t)$

Pf.

$v = \|\vec{v}\|$

unit in direction of $\vec{v}$

$$\vec{a}(t) = \frac{d}{dt} \vec{v}(t) = \frac{d}{dt}\left[\frac{ds}{dt} T(t)\right]$$

$$= \frac{d^2s}{dt^2} T(t) + \frac{ds}{dt} \underbrace{T'(t)}_{\frac{ds}{dt} K N(t)}$$

$$= \frac{d^2s}{dt^2} T(t) + \left(\frac{ds}{dt}\right)^2 K N(t)$$

$$\boxed{\vec{a}(t) = \frac{d^2s}{dt^2} T(t) + v^2 K N(t)}$$