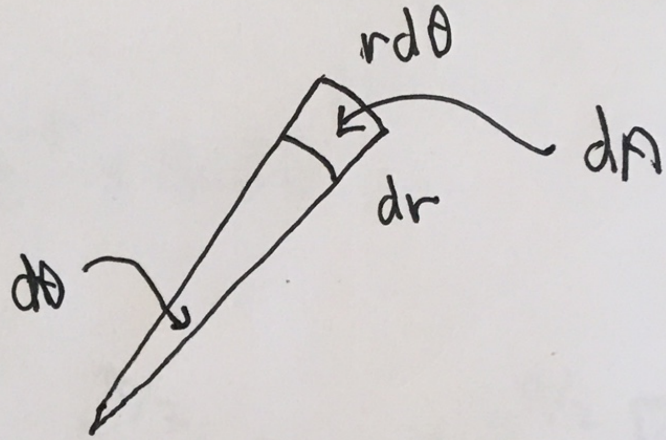
§15.3

- Polar Coordinates –

Evaluate $\iint_{R_{xy}} f(x,y)\,dA$ in Polar Coords

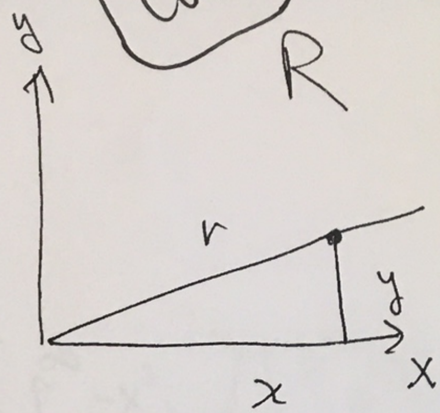
$dA = r\,dr\,d\theta$



$$\iint_{R_{xy}} f(x,y)\,dA = \iint_{R_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

"r is the amplication factor for area in going from polar coords to (x,y)-coords"

<u>Ex</u> Evaluate: $\iint\limits_R e^{-x^2-y^2}\,dA$ Polar coords



R_{xy} : $0 \le x \le \infty$, $0 \le y \le \infty$

<u>Soln</u> $R_{r\theta}$ $0 \le \theta \le \pi/2$ $\quad 0 \le r < \infty$

$x^2 + y^2 = r^2$

$$\iint\limits_{R_{xy}} e^{-x^2-y^2}\,dA = \iint\limits_{R_{r\theta}} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\infty} re^{-r^2}\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{2}\,d\theta = \frac{1}{2}\theta\Big]_0^{\pi/2} = \boxed{\pi/4}$$

$$\int_0^{\infty} re^{-r^2}\,dr = \lim_{a\to\infty}\left\{\int_0^a re^{-r^2}\,dr = -\frac{1}{2}\int_{r=0}^{r=a} e^u\,du = -\frac{1}{2}e^{-r^2}\Big]_0^{\infty} = -\frac{1}{2}\left(e^{-a^2}-1\right)\right\}$$

$u = -r^2$

$du = -2r\,dr$

$= \frac{1}{2}$

Ex: Evaluate $\int_0^\infty e^{-x^2}\,dx$

Consider $I = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy$

$$= \int_0^\infty e^{-y^2} \underbrace{\left\{ \int_0^\infty e^{-x^2}\,dx \right\}}_{\text{a constant}} dy = \left\{ \int_0^\infty e^{-x^2}\,dx \right\} \left\{ \int_0^\infty e^{-y^2}\,dy \right\}$$

$$= \left\{ \int_0^\infty e^{-x^2}\,dx \right\}^2 = I$$

$$\Rightarrow \boxed{\int_0^\infty e^{-x^2}\,dx = \sqrt{I}}$$

Conclude - it suffies to show $\int_0^\infty \int_0^\infty e^{-r^2}\,dx\,dy = \frac{\pi}{4}$

i.e. then $\sqrt{I} = \frac{\sqrt{\pi}}{2}$

§15.7

Gaussian Distribution

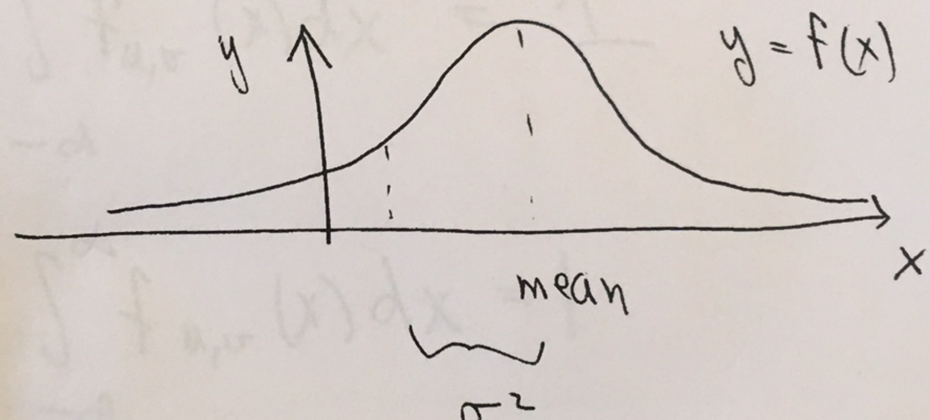
$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \equiv f_{\mu,\sigma}(x)$$

$x = \mu$ mean

$\sigma^2 =$ Variance

$\sigma \equiv$ standard Deviation

Graph:



Thm: The average of N outcomes of a random variable (appropriately rescaled) always tends to $f_{\mu,\sigma}$ for some μ & some σ

"Central Limit Theorem"

In the theory of probabity (Kolmagorov)

$$\text{Prob}\{x \in [a,b]\} = \int_a^b f_{\mu,\sigma}(x)\,dx.$$

For the theory to make sense, we need:

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = 1$$

Prove: $\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = 1$.

Proof:

- Change Variables

$$\cancel{f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2}}$$

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2} dx$$

$$v = \frac{x-\mu}{\sqrt{2}\,\sigma}$$

$$dv = \frac{dx}{\sqrt{2}\,\sigma}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-v^2} du = \frac{2}{\sqrt{\pi}} \underbrace{\int_{0}^{\infty} e^{-v^2} dv}_{\frac{\sqrt{\pi}}{2}}$$

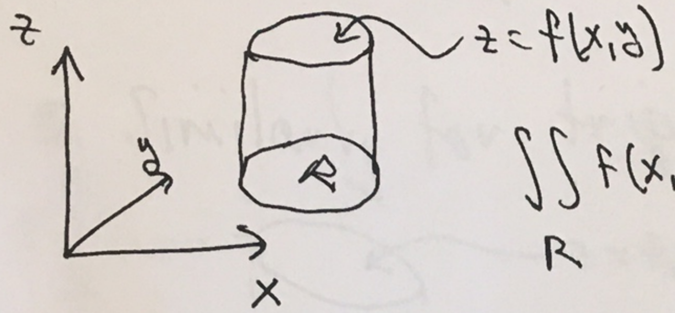
~~• Conclude: it suffices to prove that~~ $\cancel{\int_{-\infty}^{\infty} e^{-u^2} du = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$

~~Conclude: It suffices to prove:~~ $\int_{0}^{\infty} e^{-v^2} dv = \frac{\sqrt{\pi}}{2}$

Triple Integrals Intro

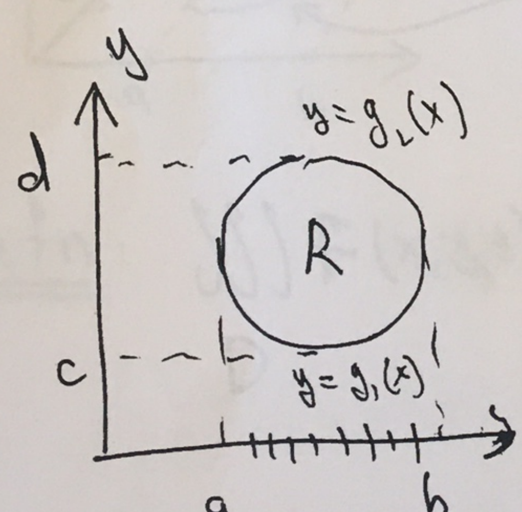
15.4

- Recall – Double integral $z = f(x,y)$



$$\iint_R f(x,y)\,dA = \text{"Vol under graph of } f \text{ ~~below~~ above } R\text{"}$$

- To define & evaluate you only need to look at R in xy-plane



$$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \sum f(x_i, y_i)\Delta x$$

Riemann Sum

Notation— $dA = dx\,dy = dy\,dx$

$x_i = a + i\Delta x \quad \Delta x = \frac{b-a}{N}$

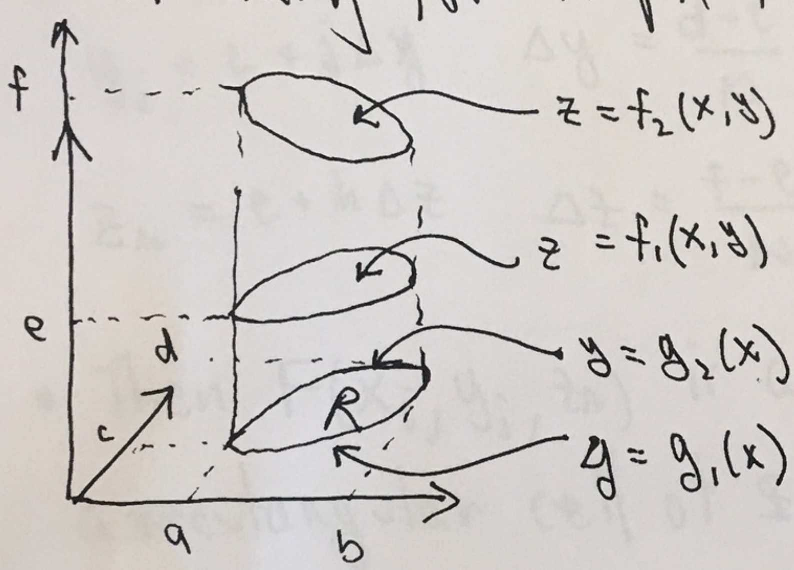
$y_j = c + j\Delta y \quad \Delta y = \frac{d-c}{N}$

You can view ↑ as a "sum of areas of rectangles weighted by..."

• To evaluate, iterate the integral —

$$\iint_R f(x,y)\,dA = \int_a^b A(x)\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

☒ Similarly for triple integrals: $w = F(x,y,z)$



$D \equiv$ Region in xyz-space bdd by R, betw f_1 & f_2.

Defn: $$\iiint_D F(x,y,z)\,dV = \lim_{N\to\infty} \sum F(x_i, y_j, z_k)\,\Delta x\,\Delta y\,\Delta z$$

sum over $(x_i, y_j, z_k) \in D$

•

Procedure to define integral —

- Divide $[a,b]$, $[c,d]$, $[e,f]$ into intervals of length $\Delta x, \Delta y, \Delta z$ by

$$x_i = a + i\Delta x \qquad \Delta x = \frac{b-a}{N}$$

$$y_j = c + j\Delta y \qquad \Delta y = \frac{d-c}{N}$$

$$z_h = e + h\Delta z \qquad \Delta z = \frac{f-e}{N}$$

- Then $F(x_i, y_j, z_h)$ is a point within a ~~s~~ rectangular cell of size

$$\Delta V_{ijh} = \Delta x \, \Delta y \, \Delta z$$

- $$\iiint_D F(x,y,z)\, dV = \lim_{N\to\infty} \sum_{(x_i, y_j, z_h)\in D} F(x_i, y_j, z_h)\, \Delta x \Delta y \Delta z$$

"view the Riemann sum as a weighted volume of D i.e. a sum of volumes of Rectangles weighted ~~+..~~ ~~by~~ values of F in each cell the formula for iterated integrals

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☒ We ~~get this~~ from the Definition —

$$\iiint_D f(x,y,z)\,dV = \lim_{N\to\infty}\left\{\sum_{i,j,k \text{ in } D} f(x_i, y_j, z_k)\,\Delta x \Delta y \Delta z\right\}$$

$$\lim_{m\to\infty}\{\ \} = \lim_{N\to\infty}\sum_{i,j}\underbrace{\underbrace{\left[\sum_{k} f(x_i, y_j, z_k)\,\Delta z\right]}_{\int_{f_1(x_i,y_j)}^{f_2(x_i,y_j)} f(x_i,y_j,z)\,dz}}_{A(x_i,y_j)}\Delta x \Delta y$$

fix x_i, y_j

$$= \lim_{N\to\infty}\sum_{i,j\in R_{xy}} A(x_i,y_j)\,\Delta x\Delta y = \iint_{R_{xy}} A(x,y)\,dA$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)}\left[\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\right] dy\,dx \quad \checkmark$$

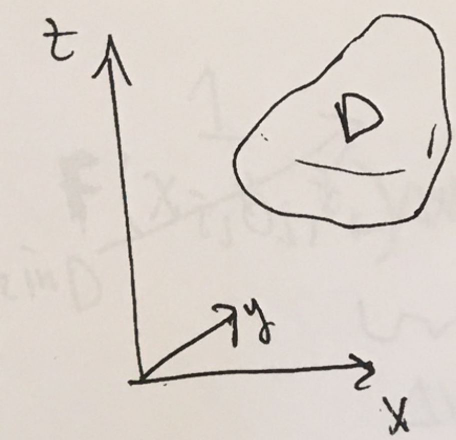
Given density $= \delta(x,y,z) = \frac{\text{mass}}{\text{vol}}$

(5) From the Defn. of Integral we use the same argument to define ① Total Mass M ② Average Density ③ Center of mass $(\bar{x}, \bar{y}, \bar{z})$ ④ KE_L of rotation. For this —

Assume 3-D object D

Assume density given

$$\delta(x,y,z) = \frac{\text{mass}}{\text{vol}} \text{ at } (x,y,z)$$



①
$$M = \lim_{N\to\infty} \sum_{i,j,k \text{ in } D} \underbrace{\delta(x_i, y_j, z_k)\,\Delta x \Delta y \Delta z}_{\Delta M_{ijk}}$$

$$= \iiint_D \delta(x,y,z)\, dV$$

$\equiv$ "Total mass of D"

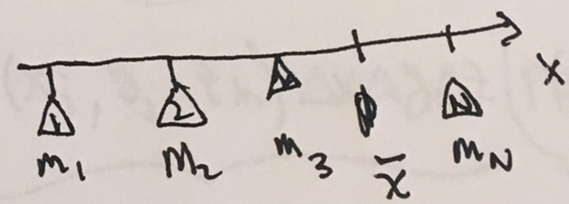
2 Average Density $= \dfrac{\text{Total Mass}}{\text{Vol}}$

$$= \frac{\iiint\limits_D \delta(x,y,z)\,dv}{\iiint\limits_D dv}$$

$$\lim_{N\to\infty} \sum_{ijk \text{ in } D} \overset{1}{\cancel{F}}(x_i,y_j,z_k)\underbrace{\Delta x\Delta y\Delta z}_{\Delta V_{ijk}}$$

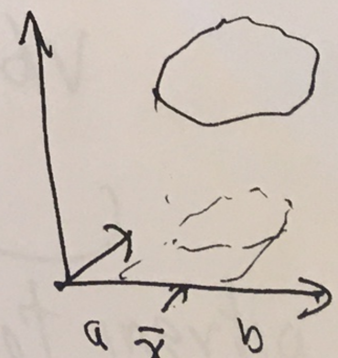
3 Center of mass:

for finite # masses:



balance pt $\bar{x}$: $\displaystyle\sum_{i=1}^{N}(\bar{x}-x_i)m_i = 0$

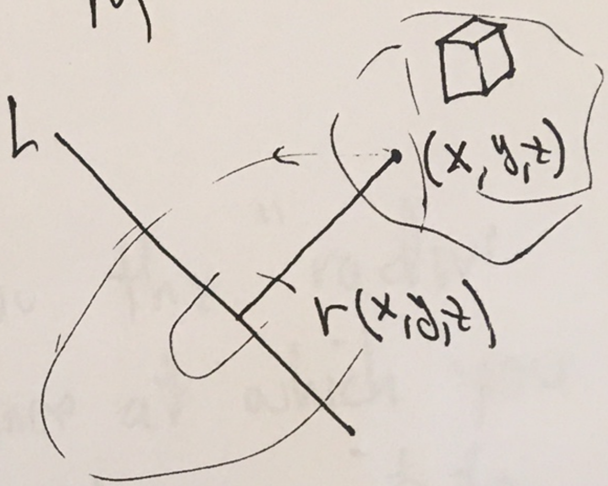
solve for $\displaystyle\bar{x} = \frac{\sum_{i=1}^{N} x_i m_i}{\sum_{i=1}^{N} m_i}$



$$\therefore\ \bar{x} = \frac{\displaystyle\lim_{N\to\infty}\sum_{i,j,k \text{ in } D} x_i\,\delta(x_i,y_j,z_k)\Delta x\Delta y\Delta z}{\text{Total mass}}$$

$$\bar{x} = \frac{1}{M}\iiint_D \underbrace{x\,\delta(x,y,z)\,dV}_{M_{xy}} \equiv \frac{M_{yz}}{M}$$

$$\bar{y} = \frac{M_{xz}}{M} \qquad \bar{z} = \frac{M_{xy}}{M}$$



④ KE_L

$\left(KE = \frac{1}{2}mv^2\right)$

$$KE_L = \lim_{\substack{N \to \infty \\ (x_i, y_j, z_k) \text{ in } D}} \sum \frac{1}{2} \underbrace{\delta(x_i, y_j, z_k)\,\Delta x \Delta y \Delta z}_{\Delta M_{ijk}} \underbrace{\left(r(x_i, y_j, z_k)\right.}_{V_{ijk}}$$

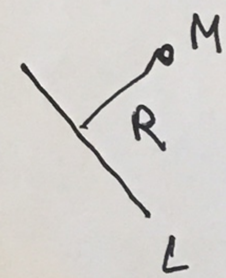
$$= \frac{1}{2}\omega^2 \underbrace{\iiint_D r(x,y,z)^2\,\delta(x,y,z)\,dV}_{I_L \equiv \text{"Mom of inertia of D about L"}}$$

5 Radius of Gyration — "The radi:

$$\frac{1}{\cancel{2}} M (R\cancel{\omega})^2 = \frac{1}{\cancel{2}} I_L \cancel{\omega^2}$$

$$R = \sqrt{I_L / M}$$

Ex: Find a formula for the "radius of gyration", the distance at which you could put All the mass at a point to get the ~~Eof~~ same KE



Soln: $KE = \frac{1}{2} M v^2 = \frac{1}{2} M (R\omega)^2$ ← if all mass at R

$$= \frac{1}{2} I_L \omega^2$$

Solve for R: $R = \sqrt{I_L / M}$