Math 21D ①
Lecture
Friday Oct 8
2021

§ 15.5 Triple Integrals –

D a volume in (x,y,z)-space $\mathbb{R}^3$

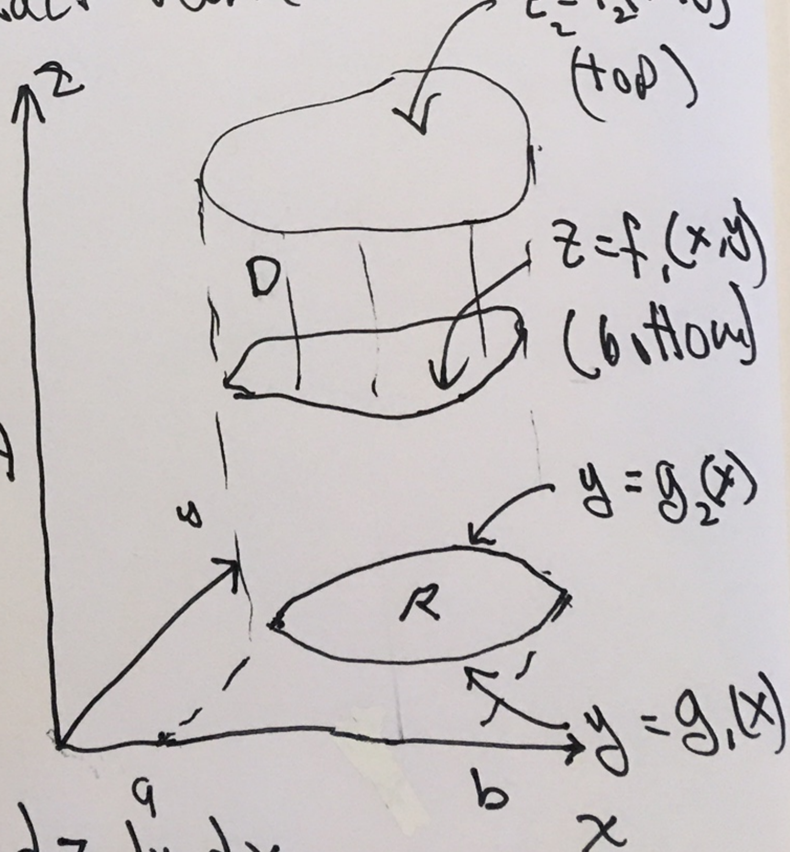
- $$\iiint_D F(x,y,z)\,dV = \lim_{N\to\infty} \underbrace{\sum_{(x_i,y_j,z_k)\in D} F(x_i,y_j,z_k)\,\Delta x\,\Delta y\,\Delta z}_{\text{3-d Riemann Sum}}$$

- Iterate to get exact value

$$\iiint_D F(x,y,z)\,dV$$

$$= \iint_R \underbrace{\left(\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\right)}_{A(x,y)} dA$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$



We have -

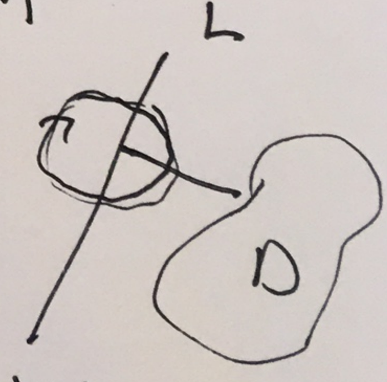
$$Vol(D) = \iiint_D dV$$

$$Mass = M = \iiint_D \delta(x,y,z)\, dV$$

$$\bar{x} = \frac{1}{M}\iiint_D x\,\delta(x,y,z)\, dV = \frac{M_{yz}}{M}$$

$$\bar{y} = \frac{1}{M}\iiint_D y\,\delta(x,y,z)\, dV = \frac{M_{xz}}{M}$$

$$\bar{z} = \frac{1}{M}\iiint_D z\,\delta(x,y,z)\, dV = \frac{M_{xy}}{M}$$



$$KE_I = \frac{1}{2} I_L \omega^2 \qquad \omega = \frac{d\theta}{dt}$$

$$I_L = \iiint_D r(x,y,z)^2\, \delta(x,y,z)\, dV$$

$\equiv$ moment of inertia about axis I_L

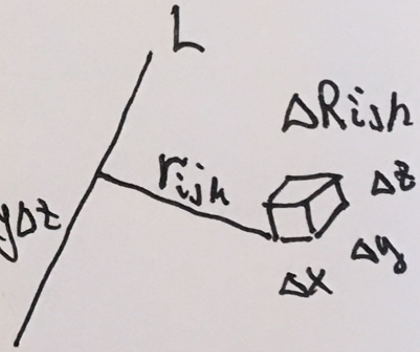
Ex: Derive a formula for the radius of gyration R of a block of constant density δ & sides of length

$$0 < a < b < c$$

about an an axis along <u>shortest</u> edge.

Soln: First recall derivation of formula for KE of rotation –

$$\Delta KE_{ijk} = \frac{1}{2}\Delta M_{ijk} V_{ijk}^2 = \frac{1}{2}\delta(x_i y_j z_k)\Delta x \Delta y \Delta z \cdot r_{ijk}^2 w^2$$



$$KE = \lim_{N\to\infty}\left\{\sum_{ijk}\Delta KE_{ij} = \sum_{(x_i y_j z_k)\in D}\frac{1}{2} r_{ijk}^2\,\delta(x_i y_j z_k)\Delta x \Delta y \Delta z\right\} \cdot w^2$$

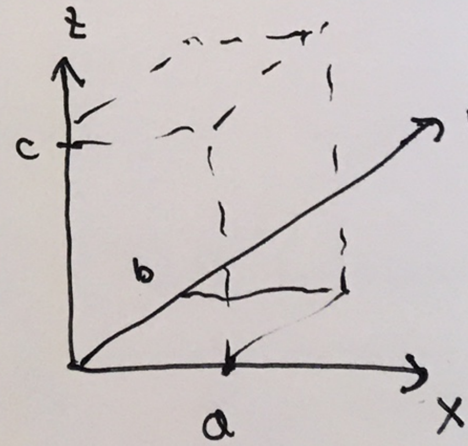
$$= \frac{1}{2} I_L w^2 \qquad I_L = \iiint_D r(x,y,z)^2\,\delta(x,y,z)\,dV$$

The radius of gyration is dist at which you put all the mass to get the same total KE:

I.e. R satisfies $\frac{1}{2}M\underbrace{R^2\omega^2}_{v^2} = \frac{1}{2}I_L\omega^2$

Solve for R $\quad R^2 = \frac{I_L}{M} \quad \boxed{R = \sqrt{\frac{I_L}{M}}}$

- For our problem D is box of sides a, b, c



Shortest side is a, so

L = x-axis

r = distance from x-axis

$= \sqrt{y^2 + z^2}$

$$M = \iiint_D \delta(x,y,z)\,dV = \int_0^c\int_0^b\int_0^a \underbrace{\delta(x,y,z)}_{\delta = \text{const}}\,d\cancel{x}\,dy\,dz$$

$$= \delta \int_0^c\int_0^b\int_0^a dx\,dy\,dz$$

$$= \delta \int_0^c\int_0^b x\Big]_0^a\,dy\,dz = \delta\int_0^c\int_0^b a\,dy\,dz$$

$$= \delta a \int_0^c y\Big]_0^b\,dz = \delta a\int_0^c b\,dz$$

$$= \delta ab \int_0^c dz = \delta ab\; z\Big]_0^c = \delta abc$$

$$I_L = \int_0^c\int_0^b\int_0^a \underbrace{(y^2+z^2)}_{r^2}\,\delta\,dx\,dy\,dz$$

$$I_L = \delta \int_0^c \int_0^b y^2+z^2 \ \underbrace{x\Big]_0^a}_{a} dy\,dz$$

$$= \delta a \int_0^c \int_0^b y^2+z^2 \, dy\,dz$$

$$= \delta a \int_0^c \underbrace{\int_0^b y^2 \, dy}_{\frac{y^3}{3}\Big]_0^b} dz + \delta a \int_0^c \underbrace{\int_0^b z^2 \, dy}_{z^2 \cdot b} dz$$

$$= \delta a \frac{b^3}{3} \int_0^c dz + \delta a b \underbrace{\int_0^c z^2 \, dz}_{\frac{z^3}{3}\Big]_0^c}$$

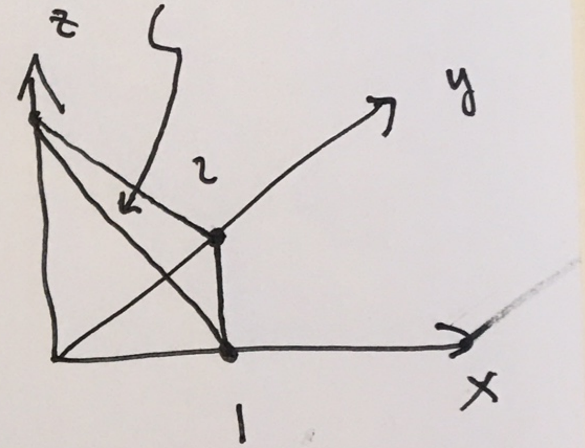
$$= \delta \frac{ab^3 c}{3} + \delta a b \cdot \frac{c^3}{3} = \frac{1}{3} abc(b^2+c^2)\delta$$

Conclude: $R = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{\frac{1}{3}abc(b^2+c^2)\cancel{\delta}}{abc\,\cancel{\delta}}}$

$= \sqrt{\frac{b^2+c^2}{3}}$ ✓

Ex: Set up the iterated integrals for the center of mass of a tetrahedron B with sides of length 1m, 2m, 3m &

$$\delta = e^{xyz} \frac{kg}{m^3} \qquad z = f(x,y)$$



Soln Graph

$$\bar{x} = \frac{\iiint_D x e^{xyz}\, dv}{M}$$

$$\bar{y} = \frac{1}{M}\iiint_D y\, e^{xyz}\, dv \qquad \bar{z} = \frac{1}{M}\iiint_D z e^{xyz}\, dv$$

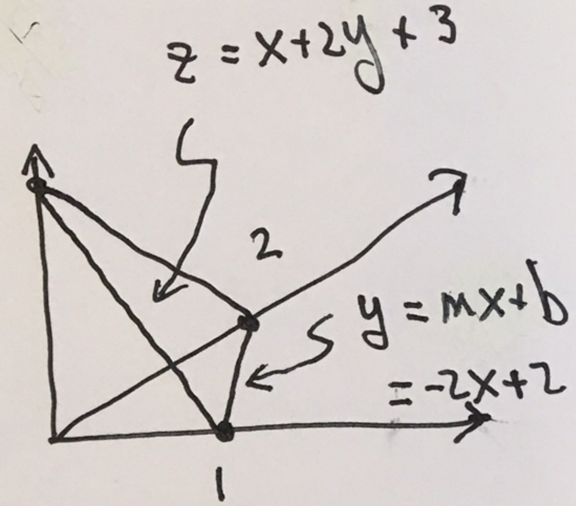
$$M = \iiint_D e^{xyz}\, dv$$

It remains to iterate the integral

$\Rightarrow$ need formula! $z = f(x,y)$ for top

The top is a <u>plane</u> so its <u>linear</u>

$$f(x,y) = ax + by + c$$



$f(0,0) = 3 \Rightarrow c = 3$

$f(1,0) = 0 \Rightarrow a = -3$

$f(0,2) = 0 \Rightarrow b = -3/2$

$$\boxed{f(x,y) = -3x - \tfrac{3}{2}y + 3} \quad \text{top}$$

$y = mx + b$

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$\dfrac{\Delta y}{\Delta x} = -\dfrac{2}{1}$

o in general!

$$\iiint_D F\,dV = \iint_R \int_0^{-3x-\frac{3}{2}y+3} F\,dz\,dx\,dy$$

$$\cancel{= \int_0 \int_0^{-\frac{3}{2}y+3} F\,d} \quad = \int_0^1 \int_0^{-2x+2} \int_0^{-3x-\frac{3}{2}y+3} F\,dz\,dy\,dx$$

(I did not ask you to evaluate, just set up!)