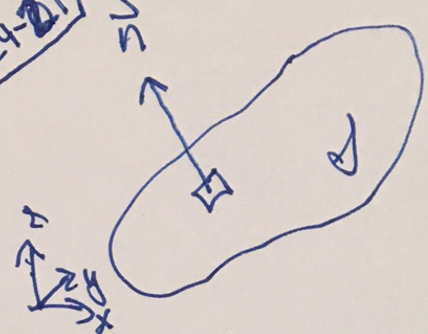
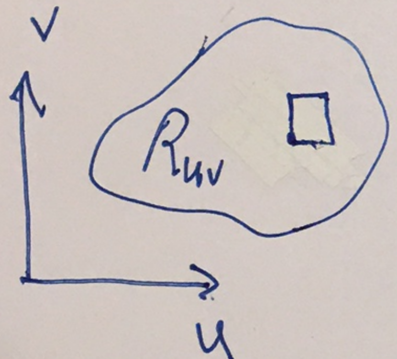
☒ Surface Integrals: $\boxed{\S 16.6}$ $\boxed{\text{Wed } 11\text{-}24\text{-}21}$

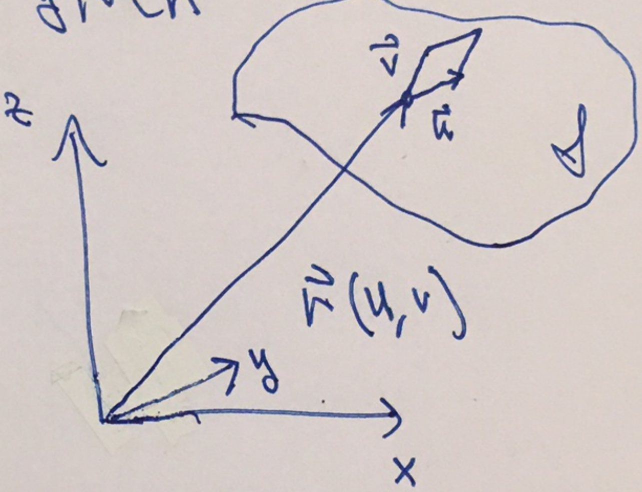
We have: $\iint_S g \, d\sigma$



element of surface area

$g(x,y,z)$ given



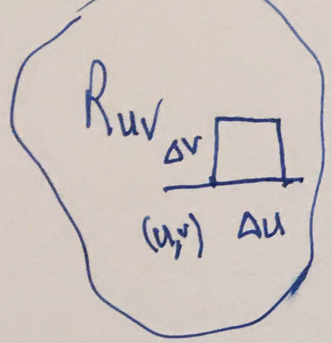


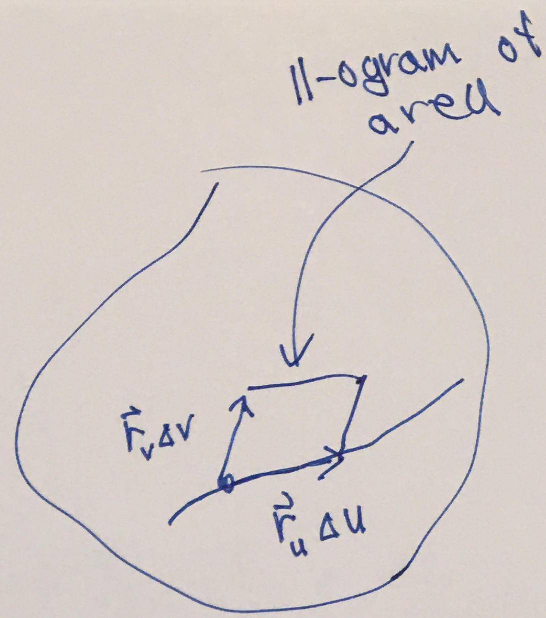
$\vec{u} = \vec{r}_u \Delta u$ $\qquad |\vec{u} \times \vec{v}| = \underbrace{|\vec{r}_u \times \vec{r}_v|}_{\text{ampl factor}} \Delta u \Delta v$

$\vec{v} = \vec{r}_v \Delta v$

$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du dv$

Lets see why





$\vec{r}_u \Delta u$ = "velocity vector"$_{\Delta u}$ $= \vec{T}_u \frac{ds}{du} \Delta u \approx \vec{T}_u ds$

$\Rightarrow$ length Δs, direction along lower side of ∥-gram

$\vec{r}_v \Delta v \approx \vec{T}_r ds \Rightarrow$ length Δs along side of ∥-gram

(28)

Gives: $$\iint_{S} g \cdot d\sigma = \iint_{R_{uv}} g(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

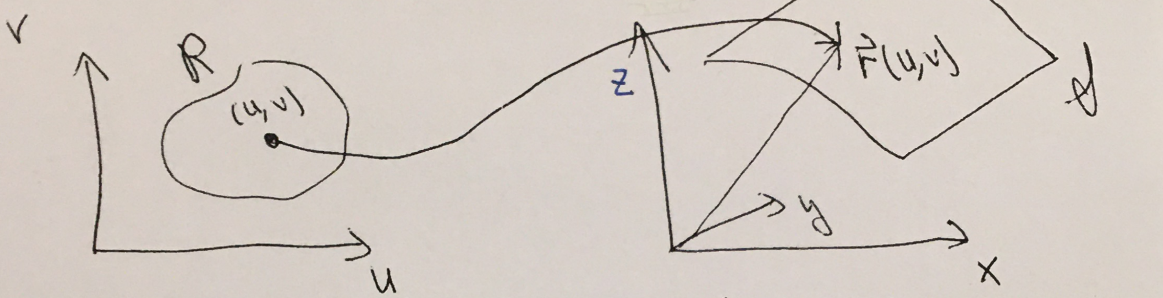
For Flux its better:

$$\iint_{S} \vec{F}\cdot\vec{n}\,d\sigma = \iint_{R_{uv}} \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{\cancel{|\vec{r}_u\times\vec{r}_v|}}\,\cancel{|\vec{r}_u\times\vec{r}_v|}\,du\,dv$$

$$= \iint_{R_{uv}} \vec{F}\cdot\vec{r}_u\times\vec{r}_v\,du\,dv$$

Surface Integrals

$$\iint_{S} g(x,y,z)\,d\sigma = \iint_{R_{uv}} g(\vec{r}(u,v))\,\underbrace{|\vec{r}_u \times \vec{r}_v|\,du\,dv}_{d\sigma}$$



$$\vec{r}(u,v) = x\,\underset{\sim}{i} + y\,\underset{\sim}{j} + z\,\underset{\sim}{k} = x(u,v)\,\underset{\sim}{i} + y(u,v)\,\underset{\sim}{j} + z(u,v)\,\underset{\sim}{k}$$

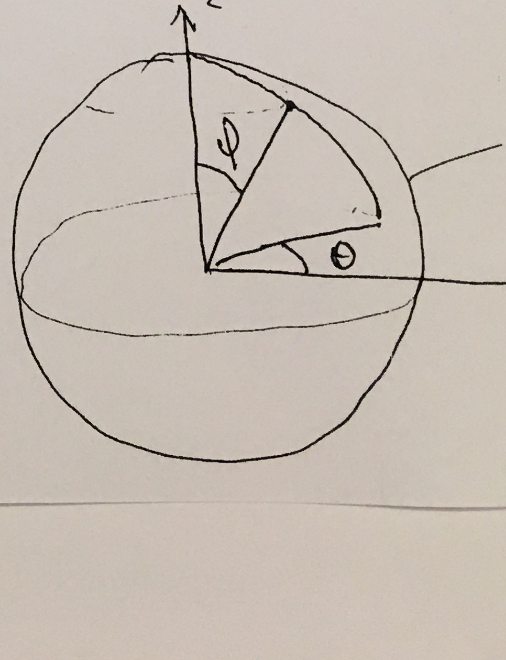
Example: Finding area on a map —

- Latitude & longitude lines put a coordinate system on the earth:

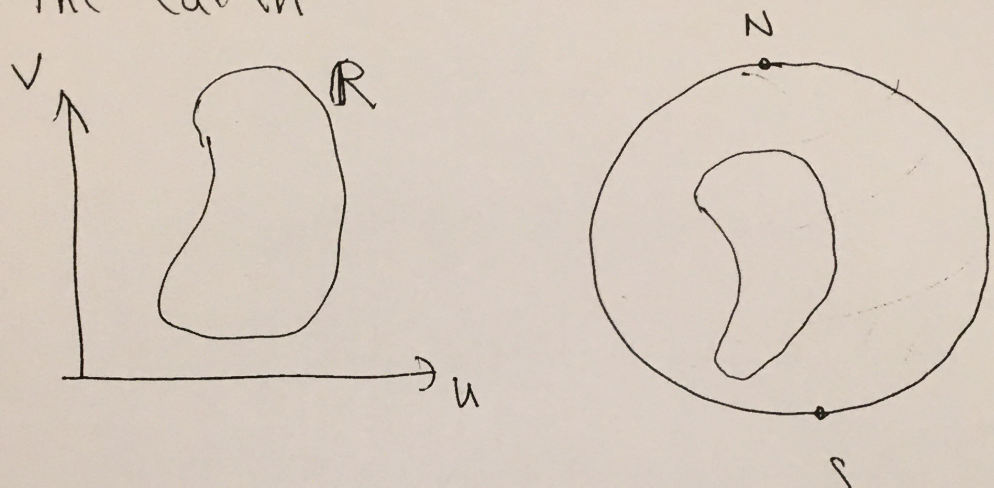
R = radius of earth $\simeq$ 4000 miles

θ = longitude

ϕ = latitude (usually measured from equator not pole)



• $u=\theta$, $v=\varphi$ puts a coordinate system on the earth -



map

<u>Thm</u>: There is no map st distance & area in uv-plane (map) = dist ~~&~~ area on globe because <u>map</u> <u>is</u> flat, sphere is <u>curved</u>

• Let $\mathcal{S}$ be a region on earth (say Africa & let R be the corresponding region on the map

(5)

Q: How do we use the map to calculate areas on the earth?

Math: A = Area of Africa $= \iint_{S} d\sigma \overset{?}{=} \iint_{R} \alpha \, du \, dv$

$d\sigma$ = area on earth $\quad \alpha$ = amplification factor

Soln: spherical coords

$$x = R\cos\theta\sin\varphi \quad y = R\sin\theta\sin\varphi \quad z = R\cos\varphi$$

$$\vec{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = R\cos\theta\sin\varphi\,\underset{\sim}{i} + R\sin\theta\sin\varphi\,\underset{\sim}{j} + R\cos\varphi\,\underset{\sim}{k}$$

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial \theta} = -R\sin\theta\sin\varphi\,\underset{\sim}{i} + R\cos\theta\sin\varphi\,\underset{\sim}{j}$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial \varphi} = R\cos\theta\cos\varphi\,\underset{\sim}{i} + R\sin\theta\cos\varphi\,\underset{\sim}{j} - R\sin\varphi\,\underset{\sim}{k}$$

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -R\sin\theta\sin\varphi, & R\cos\theta\sin\varphi, & 0 \\ R\cos\theta\cos\varphi, & R\sin\theta\cos\varphi, & -R\sin\varphi \end{vmatrix}$$

$$= \underset{\sim}{i}\left(-R^2\cos\theta\sin^2\varphi - 0\right) - \underset{\sim}{j}\left(+R^2\sin\theta\sin^2\varphi\right)$$

$$+ \underset{\sim}{k}\left(-R^2\sin^2\theta\cos\varphi\sin\varphi - R^2\cos^2\sin\varphi\cos\right.$$

$$= -R^2\left\{\cos\theta\sin^2\varphi\,\underset{\sim}{i} + \sin\theta\sin^2\varphi\,\underset{\sim}{j} + \sin\varphi\cos\varphi\,\underset{\sim}{k}\right\}$$

$$|\vec{r}_u \times \vec{r}_v|^2 = R^4\left(\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi\right)$$

$$= R^4\left(\sin^4\varphi + \sin^2\varphi\cos^2\varphi\right)$$

$$= R^4\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)$$

$$|\vec{r}_u \times \vec{r}_v| = R^2\sin\varphi \qquad \left(\geq 0 \quad 0 \leq \varphi \leq \pi\right)$$

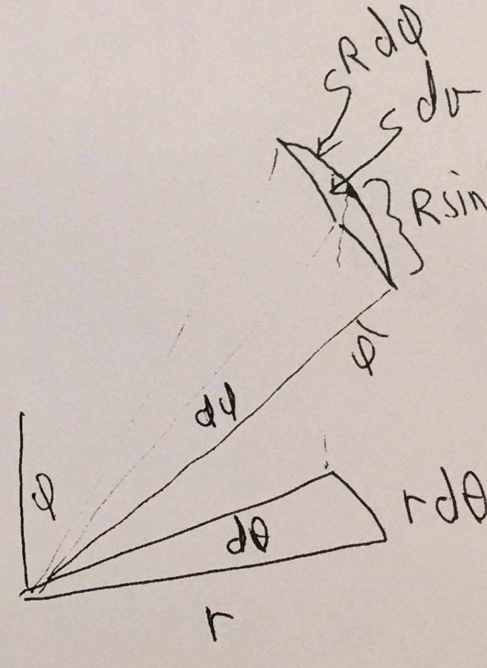
Conclude:

$$A = \iint_{\mathcal{S}} d\sigma = \iint_{\mathcal{R}} R^2 \sin\varphi \, d\varphi \, d\theta = R^2 \iint_{\mathcal{R}} \sin\varphi \, d\varphi \, d\theta$$

"Area on earth $= R^2 \times$ area on map weighted with $\sin\varphi$

- Geometrically -

$d\sigma = R^2 \sin\varphi \, d\varphi \, d\theta$ ✓



$r = R\sin\varphi$