Maxwell's Equations
Stokes Theorem
and
The Speed of Light

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1. Introduction

In the next four sections we present applications of Stokes Theorem and the Divergence Theorem. In the first application, presented here, we use them to give a physical interpretation of Maxwell's equation of electromagnetism, and then reproduce Maxwell's demonstration that his equations imply that in empty space, all components of the electric and magnetic field evolve according to the wave equation, with a speed c that can be derived by combining the electric constant and the magnetic constant in the equations. The value of c was very close to the speed of light, leading Maxwell to a rigorous proposal for Faraday's original conjecture that light actually consisted of oscillations in electromagnetic fields. This remained controversial until Heinrich Hertz generated radio waves from spinning magnets some quarter century after Maxwell's proposal. In the second application in the next section, we derive the Rankine-Hugoniot jump conditions from the weak formulation of the equations. In the third application we derive the compressible Euler equations from the physical principles of conservation of mass and momentum. And finally, in the fourth application, we introduce the heat equation and derive the maximum principle for solutions of Laplace's equation, motivating this by the condition that solutions of the heat equation decay in time to solutions of Laplace's equation, and hence the limit of the heat equation cannot support a local maximum.

2. Maxwell's Equation, the Wave Equation, and the Speed of Light

Maxwell's equations are a precise formulation Faradays laws for electromagnetism in the language of PDE's. The formulation of Maxwell's equations in the language of modern vector calculus, as first given by Gibbs, is as follows:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{4}$$

The first equation is Gauss's Law, the second Gauss's Law for Magnetism, the third is Faraday's Law of Induction, and the fourth equation is Ampere's Law with Maxwell's Correction. I.e., the last term on the RHS of (4) is the so called *magnetic induction* term, added in by Maxwell as a highly educated guess based on symmetry between $\mathbf{E}$ and $\mathbf{B}$ in (3) and (4), dimensional analysis to get the constant $\mu_0\epsilon_0$ in front of it correct, and his final justification that the resulting theory produced electromagnetic waves of speed $(\sqrt{\mu_0\epsilon_0})^{-1}$, in close agreement with the speed of light.

We comment here that Maxwell's original formulation entailed 20 complicated equations. Fortunately for us, late in the 19th century, Willard Gibbs of Yale University formulated the Stokes Theorem and Divergence Theorems in terms of the *Div* and *Curl* and used this to give the first incredibly elegant formulation (23)-(8) of Maxwell's equations. Here $\mathbf{E} = (E_1, E_2, E_3)$ is the electric field, $\mathbf{B} =$

(B_1, B_2, B_3) is the magnetic field, ρ is the charge density

$$\rho = \frac{charge}{volume},$$

and $\mathbf{J}$ is the charge flux vector, $\mathbf{J} = \rho\mathbf{u}$. The two physical constants in Maxwell's equations are ϵ_0, μ_0,

$$\epsilon_0 = \text{permittivity of free space},$$

and

$$\mu_0 = \text{permeability of free space}.$$

Here ϵ_0 is an electrical constant and μ_0 a magnetic constant that can be determined by experiments involving charges, currents and magnets alone, and reasonably good approximations for these constants were known in Maxwell's time.

To "solve" Maxwell's equations, one must find functions $E(\mathbf{x}, t)$, $B(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$ and $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, for each $\mathbf{x} = (x, y, z) \in \mathcal{R}^3$ and $t \geq 0$ that satisfy (1)-(4). In fact, the equations specify the time derivatives of $\mathbf{E}$ and $\mathbf{B}$, but to close the equations (1)-(4) one would need to augment (1)-(4) with equations for the evolution of ρ and $\mathbf{u}$, the fluid of charges in motion. For example, one could couple Maxwell'e equation to the compressible Euler equations for ρ and $\mathbf{u}$. But now we are interested in the evolution of the pure $\mathbf{E}$ and $\mathbf{B}$ fields in empty space, our goal being to understand electromagnetic waves that propagate self-consistently, un-influenced by sources of charges ρ or currents $\mathbf{J}$. For this case, set $\rho = 0$, making $\mathbf{J} = \rho u = 0$ as well, so that (1)-(4) close up to form a self-consistent set of equations for **E** and **B**, namely,

$$\begin{aligned}\nabla\cdot\mathbf{E} &= 0, &(5)\\ \nabla\cdot\mathbf{B} &= 0, &(6)\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, &(7)\\ \nabla\times\mathbf{B} &= \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}. &(8)\end{aligned}$$

Again, the *magnetic induction* term on the right hand side of (8) was added onto Ampere's Law by Maxwell to make the equations more symmetric in **E** and **B**. Essentially, Maxwell guessed this term, and we'll see how it is required for the equations to give the wave equation in each component of the **E** and **B** fields, with speed $c = (\epsilon_0\mu_0)^{-1/2}$. Since c is very large, the term on the right hand side is very small, too small to make Faraday aware of it.

For example, once Maxwell had formulated (23)-(8) *without* the magnetic induction term, the desire for symmetry between **E** in (23) and **B** in (8) cries out for a term on the RHS of (8) proportional to $\frac{\partial\mathbf{E}}{\partial t}$. But the question then is, what constant should be taken to sit in front of this term? That is, how should Maxwell guess this constant when he didn't know it ahead of time? Well, equations (1)-(4) contain two fundamental constants of electricity and magnetism identified by Faraday, namely, μ_0 and ϵ_0. So the simplest correction would be with a constant determined by ϵ_0 and μ_0. But it has to have the right dimensions. Now since μ_0 sits in front of the first term on the RHS of (4), it's reasonable to write the constant in front of the missing term $\frac{\partial\mathbf{E}}{\partial t}$ as μ_0 times some other constant α_0 to be determined, but given this, how does one "guess" the value of α_0? The answer is by dimensional analysis. Namely, it has to have the right dimensions, and turns out, (we'll show this below!), ϵ_0 has exactly the right dimensions to do the trick. That's it then...the first guess for the constant in front of $\frac{\partial \mathbf{E}}{\partial t}$ on the right hand side of (8) has to be $\mu_0\alpha_0 = \mu_0\epsilon_0$ because this choice introduces no new physical constants, and the dimensions of α_0 are the same as ϵ_0. All of this cries out for the choice

$$\alpha_0 = \epsilon_0.$$

With this highly educated guess, Maxwell showed that a consequence of (23)-(8) is that each component of $\mathbf{E}$ and $\mathbf{B}$ solves the wave equation with speed $c = (\sqrt{\mu_0\epsilon_0})^{-1/2}$, (we show this below), and using the best experimental values for μ_0 and ϵ_0 known in his time, Maxwell found that c was very nearly the speed of light. Boing! Maxwell proposed his equations, and conjectured that light was actually electromagnetic radiation. But the "proof" came twenty five years later, when Hertz generated those radio waves from spinning magnets. This was one of the rare times in history when a revolutionary new technology, the radio, was instigated by a theoretical prediction that came first, and the experiments to verify it came second. We now reproduce Maxwell's argument.

• So lets first show that the dimensions of α_0 are the same as ϵ_0. First, $\mathbf{E}$ is the force experienced by a unit unit charge in an electric field, so the dimensions of $\mathbf{E}$ are force per charge,

$$[\mathbf{E}] = \frac{[\text{Force}]}{[\text{Charge}]} = \frac{ML}{QT^2}.$$

(Here we let $[\cdot]$ denote the dimensional units of what is inside, using L=length, T=time, M=mass, Q=charge, etc.) We can get the dimensions for the magnetic field from the Lorentz force law,

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \tag{9}$$

That is, once the fields $\mathbf{E}(\mathbf{x},t)$ and $B(\mathbf{x},t)$ are determined by (1)-(4), the resulting fields will accelerate a (small) charge q according to (9), where $\mathbf{v}$ is the velocity of the charge. The first term accounts for $\mathbf{E}$ being force per charge, but the additional acceleration due to $\mathbf{B}$ comes from the second term, expressing that the magnetic force is in direction perpendicular to both $\mathbf{v}$ and $\mathbf{B}$, with magnitude proportional to the charge q, and also proportional to the magnitude of $\mathbf{v}$ and $\mathbf{B}$ as well. Since every term in a physical equation has the same dimensions, we see from (9) that

$$[q\mathbf{v}\times\mathbf{B}] = [q\mathbf{E}],$$

from which it follows that

$$[\mathbf{B}] = [\mathbf{E}][v] = \frac{[\mathbf{E}]L}{T}.$$

Now if, like Maxwell, we were trying to guess the constant in front of $\frac{\partial \mathbf{E}}{\partial t} \equiv \mathbf{E}_t$ in (4), starting with the assumption that it equals $\mu_0\alpha_0$ for some constant α_0, then equating dimensions on both terms on the RHS of (4) gives

$$[\mu_0\alpha_0\mathbf{E}_t] = [\mu_0\mathbf{J}],$$

which yields

$$[\alpha_0] = \frac{[\mathbf{J}]}{[\mathbf{E}_t]} = \frac{[\rho][v]T}{[\mathbf{E}]} = \frac{[\rho]L}{[\mathbf{E}]}.$$

But from equation (1) where ϵ_0 first appears,

$$[\epsilon_0] = \frac{[\rho]}{[\nabla\times\mathbf{E}]} = \frac{[\rho]L}{[\mathbf{E}]} = [\alpha_0],$$

as claimed.

Conclude: Our proposed α_0 must have the same dimensional units as ϵ_0, namely,

$$[\alpha_0] = [\epsilon_0] = \frac{L[\rho]}{[\mathbf{E}]} = \frac{Q^2T^2}{ML^2},$$

and therefore $\alpha_0 = \epsilon_0$ is the simplest and most natural choice for Maxwell's guess as to the constant in front of the proposed $\frac{\partial \mathbf{E}}{\partial t}$ term on the RHS of (8). By this we understand how Maxwell might have guessed and proposed the magnetic induction term on the RHS of (8) as $\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$.

3. The physical interpretation of Maxwell's equations by use of Stokes Theorem and the Divergence Theorem

We next use Stokes Theorem and the Divergence Theorem to give the physical interpretation of (1)-(4). This physical interpretation, credited to Faraday, was actually the starting point of Maxwell's theory, and from this he set to put these laws into the language of PDE's, and in so doing he discovered the missing magnetic induction term in (4). In particular, he anchored Faraday's concept of the electric and magnetic *fields* by representing their components as functions and finding the PDE equations they satisfy. We now reverse the path that Maxwell took and show how one can use Stokes Theorem and the Divergence Theorem to derive the physical interpretation of (1)-(4). The argument now goes equally well both ways because of Gibb's notation for the vector calculus of Div and Curl. We begin by reviewing Stokes Theorem and the Divergence Theorem.

We first review the fundamental first and second order operators of classical physics. The three linear partial differential operators of classical physics are the Gradient=∇, the Curl=$\nabla\times$ and the Divergence=$\nabla\cdot$. That is, formally defining

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \equiv (\partial_x, \partial_y, \partial_z)\,,$$

the Gradient of a scalar function $f(x, y, z) \equiv f(\mathbf{x})$ becomes

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

and for a vector field

$$\mathbf{F}(\mathbf{x}) = (M(\mathbf{x}), N(\mathbf{x}), P(\mathbf{x})),$$

the Curl and Divergence are defined by

$$\begin{aligned} Curl(\mathbf{F}) &= \nabla \times \mathbf{F} = Det \begin{vmatrix} \mathbf{i} & \mathbf{j} & ,\mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \qquad (10) \\ &= \mathbf{i}(P_y - N_z) - \mathbf{j}(P_y - M_z) + \mathbf{k}(N_x - M_y), \end{aligned}$$

$$Div(\mathbf{F}) = \nabla \cdot \mathbf{F} = M_x + N_y + P_z. \qquad (11)$$

Note that the $Curl(\mathbf{F})$ is a vector field and the $Div(\mathbf{F})$ is a scalar function.

The three first order differential operators ∇, $Curl$, Div of classical physics are related in a remarkable way as diagrammed in Figure A. This is a snapshot way of seeing that ∇ takes scalar functions with values in $\mathcal{R}$ to vector valued functions with values in $\mathcal{R}^3$; $Curl$ takes vector valued functions with values in $\mathcal{R}^3$ to vector valued functions with values in $\mathcal{R}^3$; and Div takes vector valued functions with values in $\mathcal{R}^3$ to scalar valued functions with values in $\mathcal{R}$. The diagram indicates that when written in this order, taking any two in a row makes zero. This is really two identities:

$$Curl(\nabla f) = 0 \qquad (12)$$

$$Div(Curl\mathbf{F}) = 0. \qquad (13)$$

$$\mathcal{R} \xrightarrow{\nabla} \mathcal{R}^3 \xrightarrow{Curl} \mathcal{R}^3 \xrightarrow{Div} \mathcal{R}$$

Three first order linear differential operators of Classical Physics:

(1) Two in a row make zero.

(2) Only Curls solve Div=0, and only Gradients solve Curl=0.

Figure A

Moreover, an important theorem of vector calculus states a converse of this. Namely, if a vector field “has no singularites” (i.e., is defined and smooth everywhere in a domain with no holes), then: (1) If $Curl\mathbf{F} = 0$ then $\mathbf{F} = \nabla f$ for some scalar f; and (2) If $Div\mathbf{F} = 0$, then $\mathbf{F} = Curl\mathbf{G}$ for some vector valued function $\mathbf{G}$.

The most important *second order* operator of classical physics is the Laplacian Δ, which can be defined in terms of these by

$$\Delta f \equiv \nabla^2 f \equiv Div(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

In terms of the Lapacian we can write the three fundamental second order linear PDE’s of classical physics, the

Laplace Equation, the Heat Equation and the Wave Equation,

$$\Delta u = 0,$$
$$u_t - k\Delta u = 0,$$
$$u_{tt} - c^2\Delta u = 0,$$

respectively. We already know that the wave equation is the equation that propagates information in all directions at speed c, in the rest frame of the observer.

• Stokes Theorem and the Divergence Theorem are stated and described in the Figures 1-4 below. By these theorems we obtain a physical interpretation of the three linear first order operators of classical physics. I.e., we already know that the gradient of a function ∇f gives the magnitude and direction of steepest increase of a function f, and points perpendicular to the level surface of f through a given point. Stokes Theorem and the Divergence Theorem tell us that, at any given point, $Curl\,\mathbf{F}\cdot\mathbf{n}$ is the circulation per area around axis $\mathbf{n}$ in the vector field $\mathbf{F}$, and $Div\,\mathbf{F}$ is the flux per volume.

Stokes Theorem:

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,dA = \int\int_{\mathcal{S}} Curl\,\mathbf{F}\cdot\mathbf{n}\,dA$$

Circulation around the boundary curve $\mathcal{C}$ = Flux of the Curl though the enclosed surface $\mathcal{S}$

Figure 1

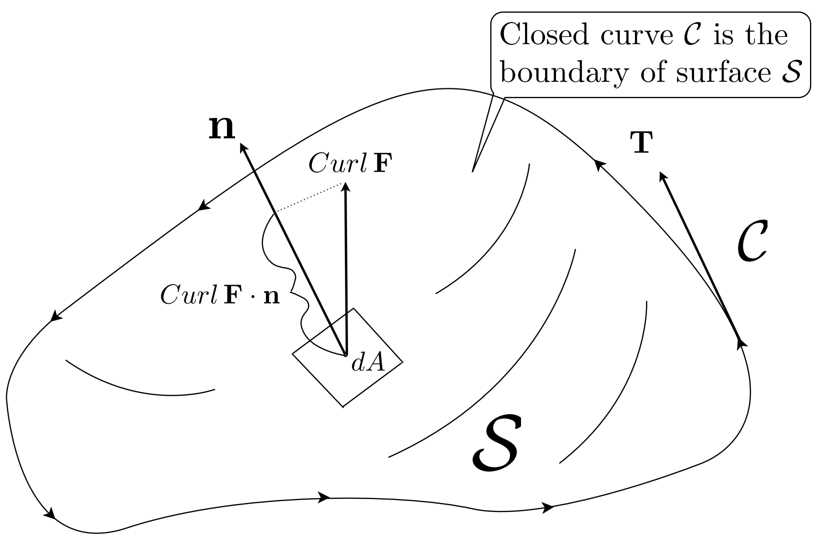


Figure 2

Divergence Theorem:

$$\int\!\!\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\, dA = \int\!\!\int\!\!\int_{\Omega} Div\,\mathbf{F}\, dV$$

$$\boxed{\begin{array}{l}\text{Flux of } \mathbf{F} \text{ through}\\ \text{closed surface } \mathcal{S} = \partial\Omega\end{array}} = \boxed{\begin{array}{l}\text{Integral of Divergence of } \mathbf{F}\\ \text{over the enclosed volume } \Omega\end{array}}$$

Figure 3

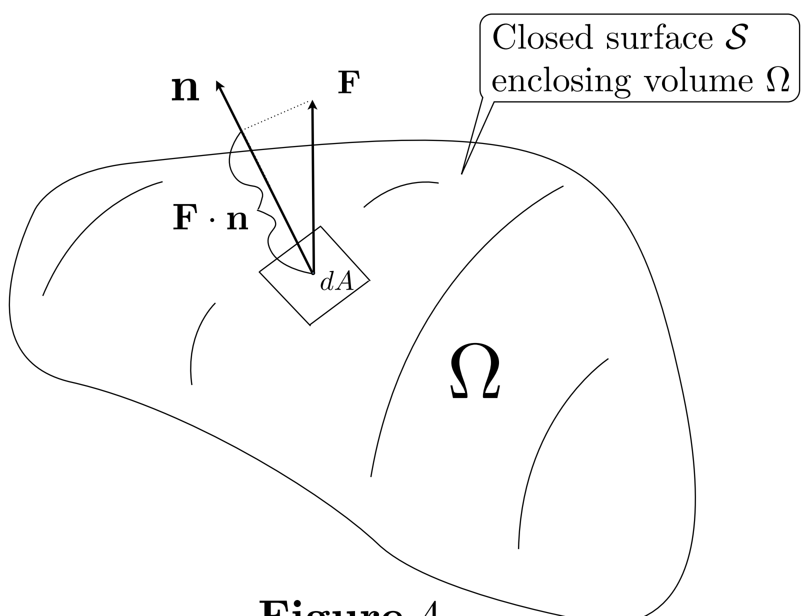


Figure 4

4. The Charge Flux Vector and the meaning of Flux

• Before applying Stokes Theorem and Divergence Theorem to Maxwell's equations, note that both the Divergence Theorem and Stokes Theorem involve the flux of a vector field through a surface. So let us recall in a paragraph the meaning of flux. The flux of a vector field $\mathbf{F}$ through a surface $\mathcal{S}$ is defined by

$$\text{Flux} = \int\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dA. \tag{14}$$

To obtain a physical interpretation of flux, consider a density $\rho \equiv \rho(\mathbf{x}, t)$ that is flowing according to a velocity $\mathbf{u}$. For example, the compressible Euler equations give the constraints on a mass density ρ moving with velocity $\mathbf{u}$ when the momentum changes are driven by the gradient of the pressure. The vector that describes the local transport of mass is then the mass flux vector $\rho\mathbf{u}$. In Maxwell's equations we have a charge density ρ moving with velocity $\mathbf{u}$, and the charge flux vector $\mathbf{J} = \rho\mathbf{u}$ also appears in the equations. Whenever you have a density ρ of "stuff per volume" being transported by velocity $\mathbf{u}$, the vector $\rho\mathbf{u}$ is the "stuff" flux vector. Assume for specificity that ρ is charge per volume. Then $\mathbf{F} = \rho\mathbf{u}$ is the charge flux vector, and its dimensions are

$$[\rho\mathbf{u}] = \frac{Q}{L^3}\frac{L}{T} = \frac{M}{L^2 T}.$$

That is, it measures charge per area time, and when dotted with the normal to a surface area dA, it gives the mass per area time passing through dA. Thus on a small area ΔA on the surface $\mathcal{S}$ oriented with normal $\mathbf{n}$,

$$\rho\mathbf{u} \cdot \mathbf{n}\Delta A = \frac{Charge}{Area \cdot Time}\Delta A = \frac{Charge}{Time}$$

moving through ΔA. Thus if the surface is discretized into a grid of N small areas of size ΔA_i as in Figure 5, the integral for the flux is approximated by the Riemann Sum

$$\begin{aligned}\int\int_{\mathcal{S}} \rho \mathbf{u}\cdot\mathbf{n}\,dA &= \lim_{|\Delta A_i|\to 0}\sum_i \rho_i \mathbf{u}_i\cdot\mathbf{n}_i\,\Delta A_i \\ &= \lim_{|\Delta A_i|\to 0}\sum_i \left\{\frac{\mathit{charge\ through}\ \Delta A_i}{\mathit{time}}\right\} \\ &= \frac{\mathit{charge\ through}\ \mathcal{S}}{\mathit{time}}, \qquad (15)\end{aligned}$$

because the sum, in the limit, is the total charge passing through $\mathbf{S}$ per time.

Conclude: The flux (14) of $\rho\mathbf{u}$=*the charge flux vector* through the surface $\mathcal{S}$ is equal to the rate at which charge is passing through $\mathbf{S}$.

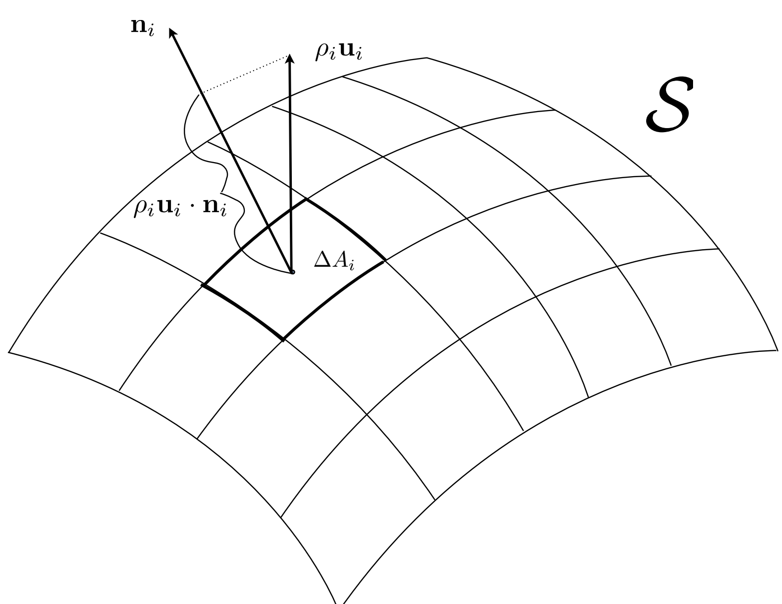


Figure 5

• We now apply Stoke's Theorem and the Divergence Theorem to Maxwell's equations (1)-(4). That is, since (1) and

(2) involve Divergence we can apply the divergence theorem over an arbitrary volume Ω, and since (1) and (2) involve Curl, we can apply Stokes Theorem over an arbitrary surface $\mathcal{S}$ with boundary $\mathcal{C}$, and then interpret the physical principles that result. Starting with (1), integrate both sides over a three dimensional surface Ω bounded by surface $\mathcal{S} = \partial\Omega$ to obtain

$$\int\int\int_{\Omega} Div\,\mathbf{E}\,dV = \int\int_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}\,dA$$

on the left hand side of (1), and

$$\int\int\int_{\Omega} \frac{\rho}{\epsilon_0}\,dV = \frac{1}{\epsilon_0}\left\{total\ charge\ in\ \Omega\right\}.$$

Equating the two we obtain Gauss Law: *The total charge within a closed surface $\mathcal{S} = \partial\Omega$ is equal to the total flux of electric field lines passing through $\mathcal{S}$, [times the constant ϵ_0.]*

Applying the same argument to equation (2) yields:

$$\int\int\int_{\Omega} Div\,\mathbf{B}\,dV = \int\int_{\mathcal{S}} \mathbf{B}\cdot\mathbf{n}\,dA = 0.$$

In words we have Gauss Law for $\mathbf{B}$: *The total flux of magnetic field lines through any closed surface $\mathcal{S} = \partial\Omega$ is zero.* This is diagrammed in Figure 6. The point is that the field lines enter with a $\mathbf{B}$ vector pointing inward through $\mathbf{S}$ and the field lines exiting point outward, so dotting with the outward normal, on average, cancels out the total flux. Note that $\mathbf{E}$ and $\mathbf{B}$ are not mass or charge or any other kind of "stuff" flux vector formed by $\rho\mathbf{u}$, so the flux of $\mathbf{E}$ or $\mathbf{B}$ is not a charge per time or mass per time flowing through $\mathcal{S}$. Moreover, the vector fields $\mathbf{E}$ and $\mathbf{B}$ do not come to us as tangent vectors to particle trajectories like the velocity $\mathbf{u}$ of a fluid of mass or charge. But we can still talk about the "field lines" as the integral curves of the electric and magnetic fields as solutions of first the order ODE's

$$\dot{\mathbf{x}} = \mathbf{E}(\mathbf{x}, t),$$

and

$$\dot{\mathbf{x}} = \mathbf{B}(\mathbf{x}, t),$$

respectively. It is very useful to have the analogy of mass and charge flux vectors in mind when we say *the total flux of* $\mathbf{B}$ *through* $\mathbf{S}$ *is zero.*

The left hand side of (3) is $Curl\, \mathbf{e}$, so to apply Stokes theorem, dot both sides of (3) by $\mathbf{n}$ and integrate over any surface $\mathcal{S}$ with normal $\mathbf{n}$ and boundary curve $\mathcal{C}$ to obtain

$$\int\int_{\mathcal{S}} Curl\, \mathbf{E} \cdot \mathbf{n}\, dA = \int_{\mathcal{C}} \mathbf{E} \cdot \mathbf{T} ds, \tag{16}$$

from the left hand side, and

$$-\frac{d}{dt}\int\int_{\mathcal{S}} \mathbf{B}\, dA. \tag{17}$$

from the right hand side of (3). Together they give

$$\int_{\mathcal{C}} \mathbf{E} \cdot \mathbf{T} ds = -\frac{d}{dt}\int\int_{\mathcal{S}} \mathbf{B}\, dA. \tag{18}$$

Here $\mathbf{T}$ is the unit tangent vector to the curve $\mathbf{C}$, related to $\mathbf{n}$ by the right hand rule, (see Figure 7). Now the line integral of $\mathbf{E}$ on the left in (18) is the circulation in $\mathbf{E}$ around the closed curve $\mathcal{C}$ because it is the sum of the components of $\mathbf{E}$ tangent to the curve weighted with respect to arclenth ds. Thus in words, Faraday's Law in the form (18) says: *The circulation in* $\mathbf{E}$ *around the boundary curve* $\mathcal{C}$ *is equal to minus the time rate of change of the flux of magnetic field lines through any surface* $\mathcal{S}$ *it bounds.*

Now the line integral for the circulation in $\mathbf{E}$ around $\mathcal{C}$ is also the integral of force times distance, which is work done by the force field $\mathbf{E}$—except $\mathbf{E}$ is really force per charge according to the Lorentz Force Law (9). So the circulation in $\mathbf{E}$ around $\mathcal{C}$ gives the work done, or energy added, to a unit charge as it moves around the loop. This is then the potential difference around a circuit $\mathcal{C}$. Conclude that Faraday's Law implies that changing magnetic field lines can create currents just like a battery can. It also tells us how currents create magnetic fields.

Finally, consider Ampere's Law (4) with Maxwell's correction term. Dotting both sides of (4) with respect to $\mathbf{n}$ and integrating over a surface $\mathcal{S}$ with normal $\mathbf{n}$ and applying Stokes Theorem as in (3) we obtain

$$\int\int_{\mathcal{S}} Curl\,\mathbf{B}\cdot\mathbf{n}\,dA = \int_{\mathcal{C}}\mathbf{B}\cdot\mathbf{T}ds, \tag{19}$$

from the left hand side, and

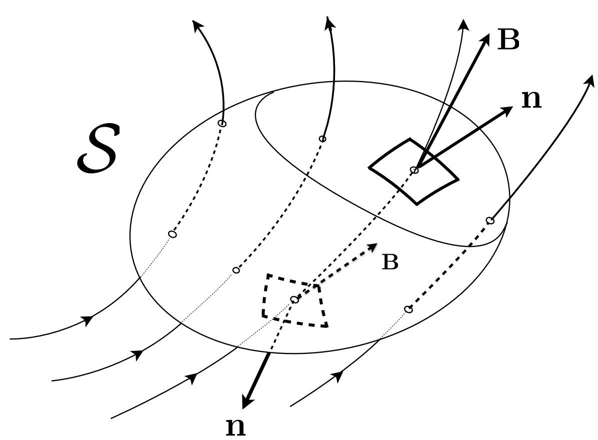
$$\mu_0\int\int_{\mathcal{S}} Curl\,\mathbf{J}\cdot\mathbf{n}\,dA + \mu_0\epsilon_0\frac{d}{dt}\int\int_{\mathcal{S}}\mathbf{B}\,dA \tag{20}$$

from the right hand side of (4). Together they give

$$\int_{\mathcal{C}}\mathbf{B}\cdot\mathbf{T}ds = \mu_0\int\int_{\mathcal{S}} Curl\,\mathbf{J}\cdot\mathbf{n}\,dA + \mu_0\epsilon_0\frac{d}{dt}\int\int_{\mathcal{S}}\mathbf{E}\,dA \tag{21}$$

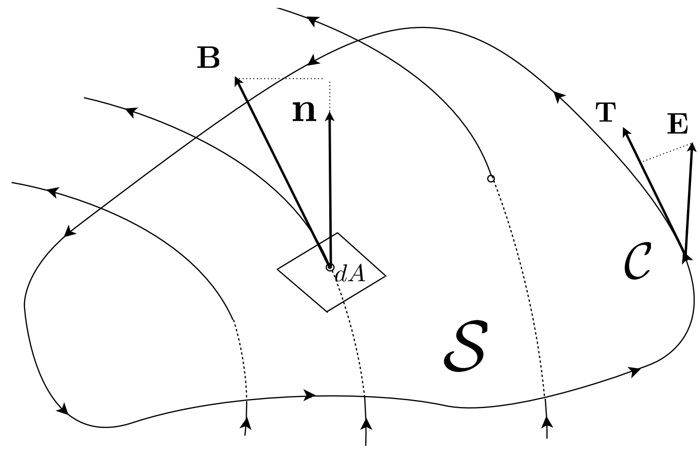
Now since $J = \rho\mathbf{u}$ is the charge flux vector, we have shown above that the first flux integral on the right hand side of (21) is the charge per time passing through $\mathcal{S}$. Thus in words, Ampere's Law in the form (21) says: *The circulation in* $\mathbf{B}$ *around the boundary curve* $\mathcal{C}$ *equal* μ_0 *times the rate at which charge is passing through* $\mathcal{S}$ *plus the rate of change of the flux of electric field lines through* $\mathcal{S}$. Maxwell's proposal was that not just moving charges could produce magnetic fields, but changing electric fields could do it as well. In the case of (23)-(8) when there are no charges or currents, this establishes a symmetry between electric and magnetic field such that changing fluxes of either one generated circulation in the other. It is natural then to wonder whether electric and magnetic fields could be self-generating. We conclude by showing that as a consequence of Maxwell's equations

(23)-(8), each component of E and B satisfy the wave equation with speed $c = (\mu_0\epsilon_0)^{1/2}$, the speed of light. Maxwell's extra term established that electromagnetic waves propagate at the speed of light, an idea that changed the course of science.



The flux of **B** going in
plus the flux out is zero

Figure 6



The potential difference ≡ the circulation in **E** around $\mathcal{C}$
equals
The time rate of change in the flux of **B**=through $\mathcal{S}$.

"Changing magnetic field lines can create electrical currents."

Figure 7

Summary: Maxwell actually reversed the steps above and derived the equations (1) – (4) from Faraday's exposition of the principles we uncovered by Stokes Theorem and the Divergence Theorem. The notation of *Div* and *Curl* came from Gibbs who reformulated Maxwell's original twenty equations into the four equations (1) – (4). We next show how the judicious choice of the constant $\mu_0\epsilon_0$ in front of the E_t term on the right hand side of (4) has the implication that (23)-(8) imply self-sustaining electromagnetic waves that propagate at the speed of light.

5. Connecting Maxwell's equations to light by means of the wave equation

We now demonstrate that solutions $\mathbf{E}(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t)$ of (23)-(8) have the property that each component of $\mathbf{E}$ and $\mathbf{B}$satisfy the wave equation with speed $c = (\mu_0\epsilon_0)^{-1/2}$. The only fact we need from vector calculus is the following identity:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \Delta\mathbf{F}. \tag{22}$$

Essentially, since $Curl\mathbf{F}$ is a vector field, and it makes sense to take the curl of it again, and the result is some expression in second order derivatives of $\mathbf{F}$, and a calculation (omitted here) yields (22).

Armed with (12)-(22) we proceed as follows: First taking the curl of both sides of (25) yields:

$$\nabla \times (\nabla \times E) = -\nabla \times \frac{\partial}{\partial t}\mathbf{B} = -\frac{\partial}{\partial t}\nabla \times \mathbf{B},$$

where we used the fact that partial derivatives commute. Using the identity (22) on the left hand side and (25) on the right hand side yields

$$\nabla(\nabla \cdot \mathbf{E}) - \Delta\mathbf{E} = -\frac{\partial}{\partial t}\mu_0\epsilon_0\frac{\partial E}{\partial t}.$$

Since $\nabla \cdot \mathbf{E} = 0$ by (23), we obtain

$$\frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{1}{\mu_0\epsilon_0}\Delta\mathbf{E} = 0,$$

where the second time derivative on the left hand side and the Laplacian on the right hand side are taken on each component of the vector $\mathbf{E}$. This then achieves half of our result: each component of $\mathbf{E}$ solves the wave equation with speed $c = (\mu_0\epsilon_0)^{-1/2}$.

Similarly for the $\mathbf{B}$ field, take the curl of both sides of (25) to obtain

$$\nabla \times (\nabla \times \mathbf{B}) = -\mu_0\epsilon_0\frac{\partial}{\partial t}\nabla \times \mathbf{E},$$

so again by (22) and (25) we obtain

$$\nabla(\nabla \cdot \mathbf{B}) - \Delta\mathbf{B} = -\frac{\partial}{\partial t}\mu_0\epsilon_0\frac{\partial \mathbf{B}}{\partial t}.$$

Since $\nabla \cdot \mathbf{B} = 0$ by (23), we obtain

$$\frac{\partial^2\mathbf{B}}{\partial t^2} - \frac{1}{\mu_0\epsilon_0}\Delta\mathbf{B} = 0,$$

thereby showing that the components of $\mathbf{B}$ also solve the wave equation with speed $c = (\mu_0\epsilon_0)^{-1/2}$. Note that from the point of view of the wave equation, the speed comes from the coefficient $\mu_0\epsilon_0$, so in principle by replacing ϵ_0 by α_0 we could achieve the wave equation for any speed c, making the dimensional analysis and the simplest choice arguments for $\epsilon_0 = \alpha_0$ very important for the discovery of electromagnetic waves.

6. Maxwells equations as evolution by the wave equation with constraints on the initial data

Now that we have shown that each component of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ propagate according to the wave equation, it makes sense to ask in what sense are Maxwell's equations (23)-(8) any more than just a complicated reformulation of the wave equation. The answer provides a powerful new way to understand how Maxwell's equations propagate the electric and magnetic fields in time. That is, we have shown that the components of $\mathbf{E}$ and $\mathbf{B}$ solve the wave equation, a second order PDE. Thus starting from the initial fields at an initial time $t = 0$, the wave equation, being *second order*, requires the values of the solution and its first time derivative as initial conditions: the values of $\mathbf{E}$, $\mathbf{B}$ as well as $\mathbf{E}_t$, $\mathbf{B}_t$ are the initial conditions at $t = 0$ that determine the solution for all time $t > 0$. (We have shown this in one dimension but it holds true in 3-space dimensions as well.) Thus, since Maxwell's equations are equations in the first derivative of $\mathbf{E}$ and $\mathbf{B}$, it follows that these are constraints that must be met by the initial conditions at $t = 0$ at the start, before the propagation by the wave equation takes over. The next theorem verifies that any solution $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ that solves the wave equation, and meets the Maxwell equations at time $t = 0$, has the property that the fields meet the Maxwell equations at every future time $t > 0$ as well. The picture then is clear: Maxwell's equations are constraints on the initial data that get propagated in time when the fields are subsequently evolved in time by the evolution of the wave equation. Since the wave equation can be thought of as the equation that propagates the solution in every direction at speed c, we see that each component of the electric and magnetic fields propagates at speed c in every direction, but all the components are constrained to be related to each other in a complicated way at each time according to the Maxwell equations (23)-(8). And the these constraints, once met by the initial data, are met at every future time as the fields evolve in time at speed c. In fact, Albert Einstein modeled the equations of general relativity after Maxwell's equations in this very way, in the sense that the equations for the curvature of spacetime are nonlinear wave equations that propagate the gravitational field at speed c, but the field must meet a first order constraint on the initial data which, once satisfied initially, is satisfied at every future time. The reason the equations of general relativity are nonlinear and the Maxwell's equations are linear even though both electromagnetic waves and gravity waves propagate at speed c in all directions according to the equations, is because in general relativity, the spacetime through which the gravity waves propagate is itself an unknown that evolves according to Einstein's equations. It takes a class in differential geometry to make precise sense of this statement.

Theorem 1. *Assume that all components of* $\mathbf{E}$ *and* $\mathbf{B}$ *evolve according to the wave equation, such that at some initial time* $t = 0$*, the initial data for these wave equations satisfies Maxwell's equations (23)-(8). Then* $\mathbf{E}$ *and* $\mathbf{B}$ *satisfy (23)-(8) at every later time* $t > 0$ *as well.*

Proof: Assume the components of **E** and **B** evolve according to the wave equation. We need only show that as a consequence of the wave equation, the time derivative of each equation (23)-(8) is zero. (Then if they start zero, they stay constant, and hence stay zero!)

$$\begin{aligned} \frac{\partial}{\partial t}\{\nabla \cdot \mathbf{E}\} &= \nabla \cdot \mathbf{E}_t \\ &= \nabla \cdot \left\{\frac{1}{\epsilon_0\mu_0}\nabla \times \mathbf{B}\right\} \\ &= \frac{1}{\epsilon_0\mu_0} Div\,(Curl\,\mathbf{B}) = 0 \end{aligned} \tag{23}$$

where we have used that mixed partial derivatives commute together with (8) to replace the time derivative of **B** with a multiple of the curl. Note how beautifully the result of Figure 13, that *Curl* followed by *Div* is always zero, comes into play. Similarly,

$$\frac{\partial}{\partial t}\{\nabla \cdot \mathbf{B}\} = 0. \tag{24}$$

For the two Maxwell equations that involve the time derivatives and the *Curl*, write

$$\begin{aligned} \frac{\partial}{\partial t}\{\nabla \times \mathbf{E} + \mathbf{B}_t\} &= \nabla \times \mathbf{E}_t + \mathbf{B}_{tt} \\ &= \frac{1}{\mu_0\epsilon_0}\nabla \times (\nabla \times \mathbf{B}) + \mathbf{B}_{tt} \\ &= \frac{1}{\mu_0\epsilon_0}\{\nabla(\nabla \cdot \mathbf{B}) - \Delta\mathbf{B}\} + \mathbf{B}_{tt} \\ &= \mathbf{B}_{tt} - \frac{1}{\mu_0\epsilon_0}\Delta\mathbf{B} = 0 \end{aligned} \tag{25}$$

because $\nabla \cdot \mathbf{B} = Div\,\mathbf{B} = 0$, because of the formula (22) for the Curl of the Curl, and because we assume **B** solves the wave equation in each component. This competes the proof of the theorem. □

Homework: Show that the Maxwell equations (6) and (8) are constant in time under the assumption that all components of **E** and **B** solve the wave equation.

7. Summary

The wave equation has played an incredible role in the history of physics. The equation was first proposed by D'Alembert in the late 1740s as a model for the vibrations of a string, some seven decades after Newton's Principia in 1687. His idea was that if $u(x,t)$ is the displacement of a string from equilibrium, then u_{tt} represents the continuum version of acceleration in Newton's force law $F = ma$, and for weak vibrations of a string, the force should be proportional to the curvature of the string u_{xx}, leading to the equation $u_{tt} - c^2 u_{xx} = 0$. A few years after that, his colleague in St Petersburg, Leonhard Euler, in the court of Catherine the Great, derived the compressible Euler equations, linearized them, and discovered that the density in air supports waves that solve the same wave equation. This established the framework for the (linearized) theory of sound, with modes of vibration and linear superposition of waves, and all the rest. The picture of sound wave propagation was then clear: vibrating strings support oscillations that propagate at a constant speed according to the wave equation, and the vibrations of the string create vibrations in the air that transmit at a constant speed according to the same wave equation, but with a different speed, the speed of sound. Thus, in the mid-seventeeth century, the wave equation showed up to resolve arguably the greatest scientific problem of the day, namely, what is sound—and perhaps the great question of their day— what makes their beloved and godly violin sound so beautiful!

But in the middle of the next century, the wave equation returned to lead the way, when Maxwell formulated Faraday's Laws of electromagnetism as a system of linear PDE's. Picking up on Faraday's proposal that light might consist of self-sustaining electromagnetic waves, he crafted his equations into a form in which the components of the field satisfied the wave equation, with a speed chosen, among the constants available, to agree with the speed of light. Thus in the most natural way, the electric and magnetic fields would satisfy the wave equation and thereby support the transmission of waves at the speed of light, and his proposal that light consisted of electromagnetic waves propagating at the speed of light was as solid as the wave equation itself. Thus, a century after Euler, the wave equation showed up again to solve arguably the greatest problem of that day, namely, what is light. And the answer, confirmed by Heinrich Hertz two decades later, began the modern age with the principles sufficient to build a radio, and all else that followed.

But then there was a major problem with Maxwell's equations. Unlike the linearized sound waves obtained from the compressible Euler equations, Maxwell's equations presumed no medium for the waves to propagate within. Indeed, as we showed, the wave equation can only hold in a frame fixed with the medium of propagation, for only in this frame can all waves generated by the wave equation move at equal speed in all directions. Based on this, scientists proposed the existence of an "ether", an invisible medium that set the frame of reference for Maxwell's wave equations, something like the rest frame of the gas in Euler's equations. Then in 1905, Albert Einstein began the modern age of physics by proposing that the wave equation for electromagnetism was fundamentally correct in *every* inertial frame, and that space and time were entangled in a new way to make it so. Thus, the wave equation, resonantly tuned, so to speak, with spacetime itself, held the secret of space and time. That is, encoded within the wave equation were the spacetime transformations that preserve its form, and assuming electromagnetic waves were resonantly tuned to propagate the same in every inertial coordinate system, the very essence of spacetime was coaxed, not directly from the experiments, but from the wave equation derived from them. Thus for a third century in a row, the wave equation showed up to solve arguably the greatest scientific problem of that era.

It is interesting to note the fundamental difference between light propagation and sound wave propagation. Maxwell's equations are *linear*, but the compressible Euler equations are *nonlinear*. Thus we have seen [topics of MAT22C] that although weak vibrations in the air lead to the linear theory of sound that we all love, the nonlinearities drive strong vibrations into shock waves, leading to shock wave dissipation and the attenuation of signals by shock wave dissipation. In contrast, Maxwell's equations for light transmission are linear at the fundamental starting point. In particular, this means that the principle of linear superposition holds. As a consequence, billions of signals can be superimposed at one end of a transmission, the superposition remains intact during transmission, and the signal can then be decoded at the other end—and there are no nonlinearities and consequent shock wave dissipation present to destroy the signals. The linearity of Maxwell's equations goes a long way toward explaining why so many cell phones can operate over such long distances, all at the same time!