Name: Solutions

Student ID#: ____________

Section: ____________

Midterm Exam 1

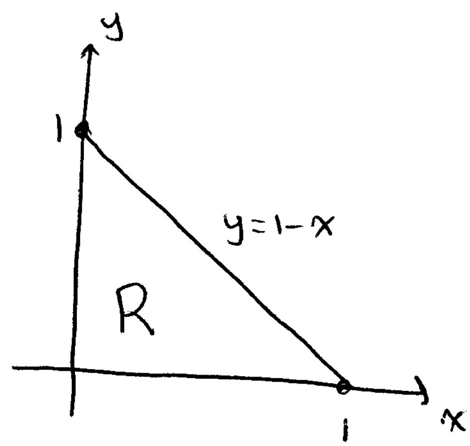
Wednesday April 25

MAT 21D, Temple, Spring 2012

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Consider a triangular metal plate with three corners (00), (1,0), (0,1) meters and constant density $\delta(x,y) = 4\ kg/m^2$. **(a)** Find the mass of the plate. (Put in units.)



$$M = \iint_R \delta\, dA = \int_0^1 \int_0^{1-x} 4\, dy\, dx$$

$$= \int_0^1 4y\Big]_0^{1-x} dx = \int_0^1 4(1-x)\, dx$$

$$= 4\int_0^1 (1-x)\, dx = 4\left(x^2 - \frac{x^2}{2}\Big]_0^1\right) = 4\left(1-\frac{1}{2}\right) = 2\ \text{kg}$$

(b) Find the center of mass $(\bar{x}, \bar{y})$. (Put in units. Note by symmetry $\bar{x} = \bar{y}$.)

$$\bar{x} = \frac{M_y}{M} = \frac{1}{2}\int_0^1 \int_0^{1-x} x \cdot 4\, dy\, dx$$

$$= 2\int_0^1 xy\Big]_0^{1-x} dx = 2\int_0^1 x(1-x)dx$$

$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\Big]_0^1\right) = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{6} = \frac{1}{3} \text{ meter}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{3}, \frac{1}{3}\right)$$

(c) Find the KE stored in rotating the plate about the y-axis at $\omega = 3$ radians/sec. (Put in the units!)

$$KE = \frac{1}{2} I_y \omega^2 = \frac{9}{2} I_y$$

$$I_y = \int_0^1 \int_0^{1-x} x^2 \cdot 4\, dy\, dx = 4 \int_0^1 x^2 y \Big]_0^{1-x} dx$$

$$= 4 \int_0^1 x^2 (1-x)\, dx = 4 \left(\frac{x^3}{3} - \frac{x^4}{4} \Big]_0^1 \right)$$

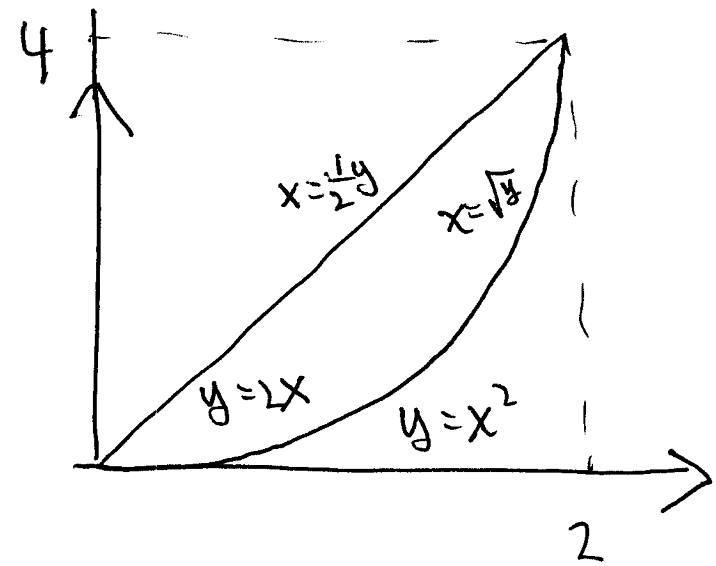
$$= 4 \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{1}{3}$$

$$KE = \frac{9}{2} \cdot \frac{1}{3} = \frac{3}{2} \underbrace{kg \cdot \frac{m^2}{s^2}}$$

units of $KE = [mv^2]$

Problem #2 (20pts): **(a)** Sketch the region of integration $\mathbf{R}_{xy}$ determined by the iterated integral

$$\int_0^2 \int_{x^2}^{2x} \sqrt{x+y}(y-2x)^2 dydx. \tag{1}$$



(b) Rewrite (2) with order of integration reversed. (Do not evaluate)

$$\int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} \sqrt{x+y}\,(y-2x)^2\,dx\,dy$$

Problem #3 (20pts): Recall that f is a probability distribution function if (i) $f(x) > 0$ for all x and (ii) $\int_{-\infty}^{+\infty} f(x)dx = 1$. In this case, the probability of finding an outcome within the interal $[a, b]$ is $Prob\{x \in [a, b]\} = \int_a^b f(x)dx$. According to the *Central Limit Theorem of Probability Theory*...No matter what the probability distribution function for a random process is, if you plot the averages of n outcomes, they will, as $n \to \infty$, fall according to the bell shaped curve called the Gaussian distribution function,

$$f(x) = ae^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

which depends on only two parameters, the mean μ and the variance σ^2. The number a must then be determined to make $\int_{-\infty}^{+\infty} f(x)dx = 1$, which is where MAT21D comes to the rescue.

(a) Evaluate the integral $I = \int_{-\infty}^{+\infty} e^{-x^2}dx$. (Hint: Polar Coordinates)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} e^{-u}du\,d\theta = \frac{1}{2}\int_0^{2\pi} -e^{-u}\Big]_0^{\infty} d\theta$$

$u=r^2$

$du=2r\,dr$

$$= \frac{1}{2}\int_0^{2\pi} d\theta = \pi \Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

(b) Prove that if $c = 1/I$, then $\int_{-\infty}^{+\infty} ce^{-x^2}dx = 1$.

Assume $c = \frac{1}{I}$. then

$$\int_{-\infty}^{\infty} ce^{-x^2}dx = \int_{-\infty}^{\infty}\frac{1}{I}e^{-x^2}dx = \frac{1}{I}\int_{-\infty}^{\infty} e^{-x^2}dx = \frac{1}{I}\cdot I = 1$$

(c) For given μ and σ, use Part (a) to find a such that $\int_{-\infty}^{\infty} a e^{-(\frac{x-\mu)^2}{2\sigma^2}} dx = 1$. (Hint: substitution)

$$\int_{-\infty}^{\infty} a e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = a\sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$u = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$du = \frac{dx}{\sqrt{2}\sigma} \qquad = a\sqrt{2}\sigma\sqrt{\pi} = 1$$

$$\Rightarrow a = \frac{1}{\sqrt{2\pi}\,\sigma}$$

Problem #4 (20pts): The mapping $x = u + v$, $y = 2v$, $z = 3w$ takes a domain D_{xyz} in (x, y, z)-space to the 3-D rectangular region $0 \le u \le 1$, $0 \le v \le 2$, $0 \le w \le 1$. Evaluate

$$\iiint_{D_{xyz}} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dxdydz.$$

$$= \iiint_{D_{uvw}} \frac{2(u+v) - 2v}{2} + \frac{3w}{3} \, |J| \, dudvdw$$

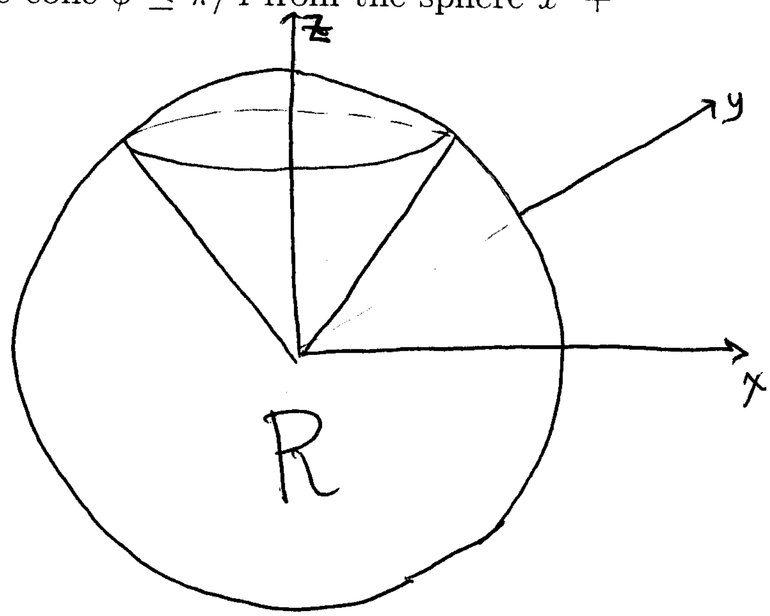
$$|J| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

$$= 6\int_0^1 \int_0^2 \int_0^1 (u+w)\, dudvdw = 6\int_0^1 \int_0^2 \frac{u^2}{2} + uw \Big]_0^1 dv\, dw$$

$$= 6\int_0^1 \int_0^2 \frac{1}{2} + w \, dvdw = 6\int_0^1 \frac{v}{2} + wv \Big]_0^2 dw = 6\int_0^1 1 + 2w\, dw$$

$$= 6\left(w + w^2 \Big]_0^1 \right) = 6 \cdot 2 = 12$$

Problem #5 (20pts): Use spherical coordinates (ρ, ϕ, θ) to find the volume of the region obtained by removing the cone $\phi \le \pi/4$ from the sphere $x^2 + y^2 + z^2 = 4$.



$$\text{Vol} = \int_0^{2\pi} \int_{\pi/4}^{\pi} \int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_{\pi/4}^{\pi} \frac{8}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} -\frac{8}{3} \cos\phi \Big]_{\pi/4}^{\pi} \, d\theta$$

$$= \int_0^{2\pi} \frac{8}{3} + \frac{4\sqrt{2}}{3} \, d\theta = \frac{8\pi}{3}(2+\sqrt{2})$$