Midterm Exam 1
Friday, January 30
MAT 21D, Temple, Winter 2015

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

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Problem #1 (20pts): (a) Sketch the region of integration $R_{xy}$ determined by the iterated integral
\[
\int_0^2 \int_{x^4}^{8x} e^{-x^2y^2} \, dy \, dx.
\]

\[ y = x^4 \quad \Rightarrow \quad x = \sqrt[4]{y} \]
\[ y = 8x \quad \Rightarrow \quad x = \frac{1}{8} y \]
\[
\int_0^{4\sqrt[4]{16}} \int_{x^4}^{8x} e^{-x^2y^2} \, dy \, dx
\]

(b) Rewrite (1) with order of integration reversed. (Do not evaluate)
Problem #2 (20pts):

(a) Integrate in spherical coordinates \((\rho, \phi, \theta)\) to find the volume of the region cut from the sphere \(x^2 + y^2 + z^2 = 9\), lying between the cones \(\phi = 0\) and \(\phi = \pi/4\).

\[
\iiint d\rho \, d\phi \, d\theta \\
0 \leq \rho \leq \frac{3}{\sin \phi} \\
0 \leq \phi \leq \pi/4 \\
0 \leq \theta \leq 2\pi
\]

\[
= 2\pi \int_0^{\pi/4} \frac{3^3}{\sin \phi} \, d\phi \\
= 18\pi \int_0^{\pi/4} \sin \phi \, d\phi \\
= 18\pi \left( [-\cos \phi]_0^{\pi/4} \right) \\
= 18\pi \left( -\frac{\sqrt{2}}{2} + 1 \right) = \sqrt{9\pi(2-\sqrt{2})}
\]
(b) Set up the integral of part (a) as an iterated integral in cylindrical coordinates \((r, \theta, z)\). (Do not evaluate).

\[
\int_{0}^{2\pi} \int_{0}^{\sqrt{9-r^2}} \int_{0}^{\sqrt{3}} r \, dz \, dr \, d\theta
\]

\[
r = \sqrt{3}, \quad \theta = \frac{\pi}{4}, \quad z = \sqrt{9-r^2}
\]

**Top:** \(z = \sqrt{9-r^2}\)

**Base:** \(z = 3\) \(r = \frac{3}{\sqrt{2}}\)

**Side:** \(\theta = \frac{\pi}{4}\) is the 45° line in \(rz\)-plane

So is curve \(z = r\)
**Problem #3 (20pts):** Let $R_{xy}$ denote a two dimensional region in the $xy$-plane bounded within the rectangle $a \leq x \leq b, c \leq y \leq d$.

(a) Give the precise definition of $\int \int_{R_{xy}} f(x,y) \, dA$ in terms of a Riemann sum making sure to define $\Delta x$, $\Delta y$, $x_i$, $y_j$ for a given positive integer $N$. Sketch $R_{xy}$ and its partition into rectangles of area $\Delta x \Delta y$.

$$\Delta x = \frac{b-a}{N} \quad \Delta y = \frac{d-c}{N}$$

$$x_i = a + i \Delta x \quad y_j = c + j \Delta y$$

$$\int \int_{R_{xy}} f(x,y) \, dA = \lim_{N \to \infty} \sum_{i,j} f(x_i,y_j) \Delta x \Delta y$$

(b) Give two reasons why the Riemann sum is only an approximation to the integral, (i.e., why we have to take a limit).

1. $f(x_i,y_j)$ is only an approx to the value of $f$ in the $ij$-th rectangle

2. Not all rectangles fit inside $R_{ij}$
(c) Assume \( R_{xy} \) is a metal plate of variable density \( \delta(x, y) \), and \( r(x, y) \) is the formula for the distance from a point \((x, y)\) to a given line \(L\). Derive our formula for the kinetic energy obtained by rotating the plate at angular velocity \(\omega\) about the axis \(L\) using the definition of an integral as a Riemann sum. (Draw a sketch, use the Newtonian relation \(KE = \frac{1}{2}mv^2\) for the kinetic energy of a point mass \(m\), and explain each step.)

\[
\text{Total } KE = \lim_{N \to \infty} \sum_{i,j} \frac{1}{2} \delta(x_i, y_j) r(x_i, y_j) \omega \Delta x \Delta y \]

Each term in sum gives approx to \(KE\) in the \(ij\)th rectangle

\[
= \frac{1}{2} \omega^2 \sum_{i,j} \delta(x_i, y_j) r(x_i, y_j)^2 \Delta x \Delta y
\]

\[
= \frac{1}{2} \iint_{R_{xy}} r(x, y)^2 \delta(x, y) \, dA \frac{\omega^2}{2} = \frac{1}{2} I_L \omega^2
\]
Problem #4 (20pts): Let $a$ be a positive constant $a > 0$. Evaluate $I = \int_{-\infty}^{\infty} e^{-ax^2} \, dx$ directly by reproducing the trick in class based on polar coordinates.

\[
\int_{-\infty}^{\infty} e^{-ax^2} \, dx = 2 \int_{0}^{\infty} e^{-ax^2} \, dx = 2I
\]

\[
\int_{0}^{\infty} \int_{0}^{\infty} e^{-a(x^2+y^2)} \, dx \, dy = \left( \int_{0}^{\infty} e^{-ax^2} \, dx \right) \left( \int_{0}^{\infty} e^{-ay^2} \, dy \right) = I^2
\]

(Also)

\[
\int_{0}^{\infty} \int_{0}^{\infty} e^{-a(x^2+y^2)} \, dx \, dy = \int_{0}^{\pi/2} \int_{0}^{\infty} e^{-ar^2} \, r \, dr \, d\theta
\]

Polar coordinates

\[
= \frac{\pi}{2} \int_{0}^{\infty} e^{-ar^2} \, r \, dr = \frac{\pi}{2} \frac{1}{2a} \int_{0}^{\infty} e^{-u} \, du = \frac{\pi}{4a} (e^0 + e^0)
\]

\[
u = ar^2
\]

\[
du = 2ar \, dr
\]

\[
I = \sqrt{\frac{\pi}{4a}} = \frac{1}{2} \sqrt{\frac{\pi}{a}} \Rightarrow \int_{-\infty}^{\infty} e^{-ax^2} \, dx = 2I = \frac{\sqrt{\pi}}{\sqrt{a}}
\]
Problem #5 (20pts): The mapping $x = 4u + v$, $y = u + v$ takes a domain $R_{xy}$ in the $(x, y)$-plane to the rectangular region $0 \leq u \leq a$, $0 \leq v \leq b$, for given positive numbers $a$ and $b$. Evaluate the integral to find a formula for the following integral in terms of $a$ and $b$:

$$I = \int \int_{R_{xy}} (x - 4y) \, dx \, dy$$

$$I = \int \int_{R_{uv}} (4u + v - 4u - 4v) \, J \, du \, dv$$

$$J = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} = 4 - 1 = 3$$

$$= \int_{0}^{b} \int_{0}^{a} \int_{a}^{4a} \, du \, dv = -9a \int_{0}^{b} v \, dv$$

$$= -9a \left[ \frac{v^2}{2} \right]_{0}^{b} = -\frac{9}{2} ab^2$$