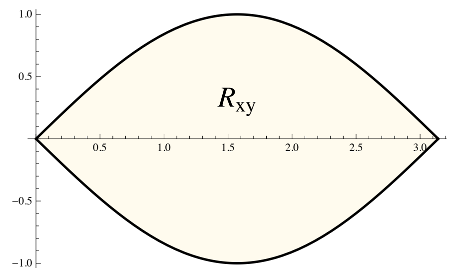
Midterm Exam 1–Solutions

MAT 21D, Temple/Romik, Spring 2016

Problem #1 (20pts): (a) Sketch and find the AREA of the region of integration $\mathbf{R}_{xy}$ determined by the following iterated integral. (You do not need to evaluate this integral.)

$$\int_0^{\pi} \int_{-\sin x}^{\sin x} \ln\left(x^2 y^4\right) dy\, dx. \tag{1}$$

Solution. $\mathbf{R}_{xy}$ is the region is bounded between the curves $y = -\sin x$ and $y = \sin x$, $0 \le x \le \pi$:



The area of $\mathbf{R}_{xy}$ is

$$\iint_{\mathbf{R}_{xy}} dA = \int_0^{\pi} \int_{-\sin x}^{\sin x} dy\, dx = \int_0^{\pi} (\sin x - (-\sin x))\, dx$$
$$= \int_0^{\pi} 2\sin x\, dx = 2(-\cos x)\Big|_{x=0}^{x=\pi} = 2(1-(-1)) = 4.$$

(b) Rewrite this iterated integral as an iterated integral with the order of integration reversed that produces the same value. (Again, you do not need to evaluate the integrals.)

$$\int_0^{1} \int_{2x^3}^{2x} \ln\left(x^2 y^4\right) dy\, dx. \tag{2}$$

Solution. The region of integration consists of pairs (x, y) that satisfy $0 \le x \le 1$, $2x^3 \le y \le 2x$. Thus as x ranges between 0 and 1, y ranges between 0 and 2, and the inequalities $2x^3 \le y \le 2x$ can be rewritten as $y/2 \le x \le (y/2)^{1/3}$, so the integral is equal to

$$\int_0^{2} \int_{y/2}^{(y/2)^{1/3}} \ln\left(x^2 y^4\right) dx\, dy.$$

Problem #2 (20pts): (a) Use polar coordinates to evaluate the integral

$$\iint_{\mathcal{R}_{xy}} e^{-r^2} dx\, dy,$$

where $\mathcal{R}_{xy}$ is the region inside the circle of radius R centered at $(0,0)$.

Solution.

$$\begin{aligned}
\iint_{\mathcal{R}_{xy}} e^{-r^2} dx\, dy &= \int_0^{2\pi} \int_0^R e^{-r^2} r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{R^2} e^{-u} \frac{du}{2}\, d\theta \quad \text{(substitution: } u = r^2\text{)} \\
&= \frac{1}{2} \int_0^{2\pi} \left(-e^{-u}\right) \Big|_{u=0}^{u=R^2} d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \left(1 - e^{-R^2}\right) d\theta = \pi \left(1 - e^{-R^2}\right).
\end{aligned}$$

(b) Since in the limit $R \to \infty$ you are integrating over all of $\mathcal{R}^2$, take the limit $R \to \infty$ in (a) to obtain a value for $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$.

Solution.

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy &= \iint_{\mathcal{R}^2} e^{-(x^2+y^2)}\, dx\, dy \\
&= \lim_{R\to\infty} \pi \left(1 - e^{-R^2}\right) = \pi(1-0) = \pi.
\end{aligned}$$

(c) Iterate the integral to show $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2$.

Solution.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \quad (\text{using } e^{a+b} = e^a e^b)$$

$$= \int_{-\infty}^{\infty} e^{-y^2}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)dy$$

$$(\text{moving a scalar } e^{-y^2} \text{ outside the integral } dx)$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$(\text{moving a scalar } \int_{-\infty}^{\infty} e^{-x^2}\,dx \text{ outside the integral } dy)$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2.$$

(d) Use (b) and (c) to evaluate the famous integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ (the *Gaussian*).

Solution. Denote $G = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. We showed that

$$G^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \pi.$$

Also clearly G is nonnegative as the integral of a nonnegative function. It follows that

$$G = \sqrt{\pi}.$$

Problem #3 (20pts): Consider a triangular metal plate with three corners $(0,0)$, $(2,0)$, $(0,2)$ meters and density $\delta(x,y) = xy\ kg/m^2$.

(a) Find the mass of the plate. (Put in units.)

Solution.

$$\begin{aligned}
M &= \int_0^2 \int_0^{2-x} \delta(x,y)\,dy\,dx = \int_0^2 \int_0^{2-x} xy\,dy\,dx \\
&= \int_0^2 x \cdot \frac{y^2}{2}\bigg|_{y=0}^{y=2-x} dx = \frac{1}{2}\int_0^2 x(2-x)^2\,dx \\
&= \frac{1}{2}\int_0^2 \left(4x - 4x^2 + x^3\right)\,dx = \frac{1}{2}\left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right)\bigg|_{x=0}^{x=2} \\
&= \frac{1}{2}\left(2\times 4 - \frac{4}{3}\times 8 + \frac{1}{4}\times 16\right)\ kg = \frac{2}{3}\ kg.
\end{aligned}$$

(b) Find the center of mass $(\bar{x}, \bar{y})$. (Put in units. Note by symmetry $\bar{x} = \bar{y}$.)

Solution.

$$\begin{aligned}
M_x &= \int_0^2 \int_0^{2-x} x\delta(x,y)\,dy\,dx = \int_0^2 \int_0^{2-x} x^2y\,dy\,dx \\
&= \int_0^2 x^2 \cdot \frac{y^2}{2}\bigg|_{y=0}^{y=2-x} dx = \frac{1}{2}\int_0^2 x^2(2-x)^2\,dx \\
&= \frac{1}{2}\int_0^2 \left(4x^2 - 4x^3 + x^4\right)\,dx = \frac{1}{2}\left(\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right)\bigg|_{x=0}^{x=2} \\
&= \frac{1}{2}\left(\frac{4}{3}\times 8 - 16 + \frac{1}{5}\times 32\right)\ kg\cdot\text{meter} = \frac{8}{15}\ kg\cdot\text{meter}.
\end{aligned}$$

The value $\bar{x}$ is therefore given by

$$\bar{x} = \frac{M_x}{M} = \frac{8/15}{2/3} = \frac{4}{5}\ \text{meters}.$$

Since $\bar{x} = \bar{y}$ as was noted in the question, the center of mass is at

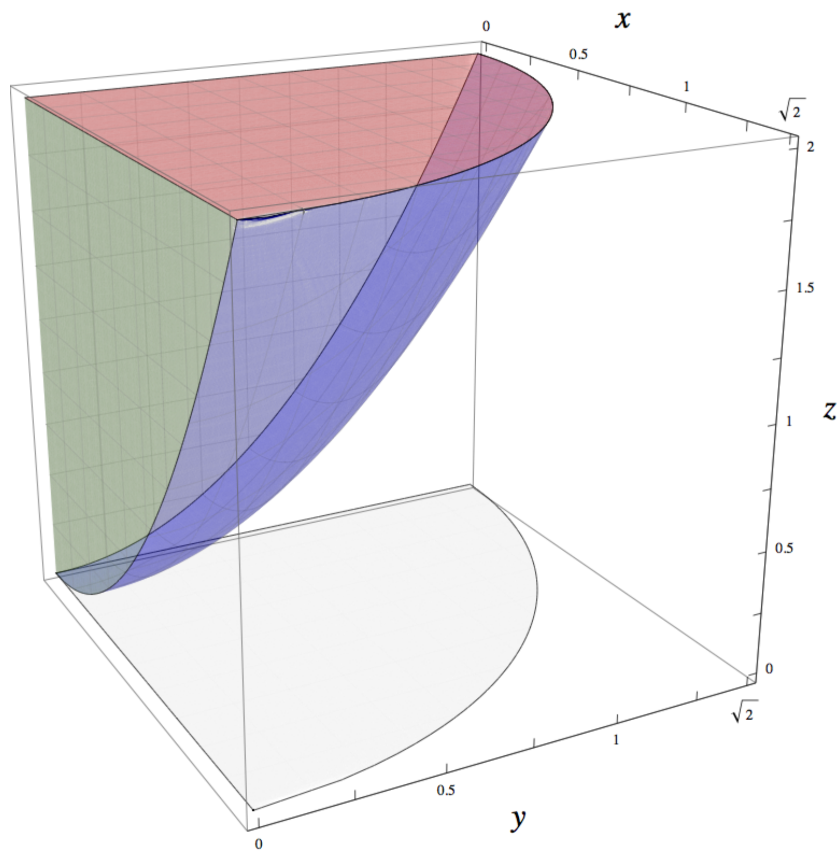
$$(\bar{x}, \bar{y}) = \left(\frac{4}{5}, \frac{4}{5}\right)\ \text{meters}.$$

Problem #4 (20pts): Evaluate $\iiint_D x\,dV$, where D is the region bounded by the planes $x=0$, $y=0$ and $z=2$, and the surface $z=x^2+y^2$ and lying in the quadrant $x\ge 0$, $y\ge 0$. Sketch the region.

Solution.

$$\begin{aligned}\iiint_D x\,dV &= \int_0^{\sqrt{2}}\left[\int_0^{\sqrt{2-x^2}}\left(\int_{x^2+y^2}^{2} x\,dz\right)dy\right]dx\\ &= \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} x(2-x^2-y^2)dydx\\ &= \int_0^{\sqrt{2}} x\left[(2-x^2)^{3/2}-\frac{(2-x^2)^{3/2}}{3}\right]dx\\ &= \int_0^{\sqrt{2}} \frac{2x}{3}(2-x^2)^{3/2}dx = \left.\frac{-2(2-x^2)^{5/2}}{15}\right|_0^{\sqrt{2}}\\ &= 2\cdot\frac{2^{5/2}}{15}=\frac{8\sqrt{2}}{15}.\end{aligned}$$

Geometrically, the region is bounded between three planes and the paraboloid $z=x^2+y^2$. The intersection of the paraboloid with the plane $z=2$ (in the region $x\ge 0, y\ge 0$) is a quarter-circle $x^2+y^2=(\sqrt{2})^2$. Here is a picture:



Problem #5 (20pts): Let R_{xy} denote the rectangle

$$a \le x \le b, \ c \le y \le d.$$

Consider R_{xy} as a thin plate with uniform density δ. Derive the formula expressing the kinetic energy obtained by rotating R_{xy} at an angular velocity of ω radians per second about the x-axis as a double integral, by using the definition of the integral as a limit of Riemann sums.

Guidance. Approximate the plate as a collection of N (=some large integer) point masses each having mass $m = \delta \times \Delta A_k$ centered at points $(x_1, y_1), \ldots, (x_N, y_N)$ (where ΔA_k is the area of a small rectangle near (x_k, y_k)). Use the Newtonian relation $KE = \frac{1}{2}mv^2$ for the kinetic energy of each point mass, and interpret the total energy as a Riemann sum. Explain each step. You do not need to evaluate the double integral.

Solution. As indicated in the question, the kinetic energy of each point mass is $\frac{1}{2}mv^2 = \frac{1}{2}\delta \times \Delta A_k v^2 = \frac{1}{2}\delta \Delta A_k (\omega r_k)^2$, where r_k is the distance of the point from the axis of rotation, or in this case $r_k = y_k^2$. Thus the total kinetic energy in this approximation is

$$KE \approx \sum_{k=1}^{N} \frac{1}{2}\delta \Delta A_k \omega^2 y_k^2 = \frac{1}{2}\delta\omega^2 \left(\sum_{k=1}^{N} y_k^2 \Delta A_k \right).$$

The sum is a Riemann sum, and in the limit where the number of points grows large it converges to the limiting value

$$\iint_{R_{xy}} y^2 \, dA.$$

The total kinetic energy is therefore equal to

$$KE = \frac{1}{2}\delta\omega^2 \iint_{R_{xy}} y^2 \, dA.$$

As we discussed in class, this can also be written as

$$KE = \frac{1}{2} I_x \omega^2,$$

where I_x is the moment of inertia about the x-axis, given by

$$I_x = \iint_{R_{xy}} y^2 \delta \, dA.$$