Name:

Student ID:
Section:

1	2	3	4	5	Total

MIDTERM EXAM I
Math 21D
Temple-F08

Write solutions on the paper provided. Put your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

Problem 1. (20pts) Consider $\int\int_R \; dA$ where R is the region in the xy-plane bounded between the curves $y = x^2$ and $y = \sqrt{x}$.

(a) Sketch the region R in the xy-plane, and use this to evaluate $\int\int_R \; dA$ as an iterated integral in (x, y)-coordinates.

Soln: Area$=\int_0^1 \int_{x^2}^{\sqrt{x}} dydx = \int_0^1(\sqrt{x} - x^2)dx = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

(b) Let $x = u^2$, $y = v$. Sketch the image G of the region R in uv-coordinates, and use this to evaluate $\int\int_R \; dA$ as an iterated integral in (u, v)-coordinates.

Soln: $y = x^2 \leftrightarrow v = (u^2)^2 = u^4$; $y = \sqrt{x} \leftrightarrow v = \sqrt{u^2} = u$; and

$$|J(u,v)| = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & 0 \\ 0 & 1 \end{pmatrix} = 2u;$$

so

$$\begin{aligned} \text{Area} &= \int_0^1 \int_{u^4}^{u} J(u,v)dvdu = \int_0^1 \int_{u^4}^{u} 2udvdu = \int_0^1 2u(u - u^4)du \\ &= \left[\frac{2u^3}{3} - \frac{2u^6}{6}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}. \end{aligned}$$

(Recall: $\int\int_R f(x,y) \; dxdy = \int\int_G f(u^2, v)J(u,v) \; dudv$.)

Problem 2. (20pts) Find the moment of inertia about the z-axis of an object of constant density $\delta(x,y,z)=2$, bounded by $\rho \leq 2$, $\phi \leq \frac{\pi}{4}$, and $0 \leq \theta \leq \frac{\pi}{3}$.

$$
\begin{aligned}
\textbf{Soln}: I_z &= \int_0^{\pi/3}\int_0^{\pi/4}\int_0^2 r^2 \cdot \rho^2 \sin\phi \cdot 2\, d\rho\, d\phi\, d\theta \\
&= 2\int_0^{\pi/3}\int_0^{\pi/4}\int_0^2 \rho^4 \sin^3\phi\, d\phi\ d\theta \\
&= 2\int_0^{\pi/3}\int_0^{\pi/4}\left[\frac{\rho^5}{5}\right]_0^2 \sin^3\phi\, d\phi\ d\theta = \frac{2^6}{5}\frac{\pi}{3}\int_0^{\pi/4}\sin^3\phi\, d\phi \\
&= \frac{2^6}{5}\frac{\pi}{3}\int_0^{\pi/4}(1-\cos^2\phi)\sin\phi\, d\phi \\
&= \frac{2^6}{5}\frac{\pi}{3}(-1)\int_{\phi=0}^{\phi=\pi/4}(1-u^2)\, du, \quad (u=\cos\phi), \\
&= \frac{2^6}{5}\frac{\pi}{3}\left[\frac{\cos^3\phi}{3}-\cos\phi\right]_{\phi=0}^{\phi=\pi/4} \\
&= \frac{2^6}{5}\frac{\pi}{3}\left[\frac{2\sqrt{2}}{8\cdot 3}-\frac{\sqrt{2}}{2}-1/3+1\right] \\
&= \frac{2^6}{5}\frac{\pi}{3}\left[-\frac{10\sqrt{2}}{24}+\frac{2}{3}\right] \\
&= \frac{2^6}{5}\frac{\pi}{3}\left[\frac{16-10\sqrt{2}}{24}\right] \\
&= \frac{2^4}{5}\frac{\pi}{3^2}\left[8-5\sqrt{2}\right]
\end{aligned}
$$

Problem 3. (20pts) Let f be the *probability density* function defined for all (x,y) in the unit disk $\mathcal{R}$ (the *probability space*) by

$$f(x,y) \;=\; C\,e^{-x^2-y^2}, \qquad \mathcal{R} = \{(x,y) : x^2+y^2 \le 1\}.$$

Determine the value of constant C such that $\int\int_{\mathcal{R}} f(x,y)dA = 1$.
(Hint: Polar Coordinates!)

$$\textbf{Soln}: \int\int_{\mathcal{R}} e^{-x^2-y^2}\,dx\,dy = \int_0^1\int_0^{2\pi} re^{-r^2}\,d\theta\,dr = 2\pi\int_0^1 re^{-r^2}\,dr = 2\pi\cdot\frac{1}{2}\int_0^1 e^{-u}\,du$$

$$= \pi\left[1-e^{-1}\right] = \frac{\pi(e-1)}{e}.$$

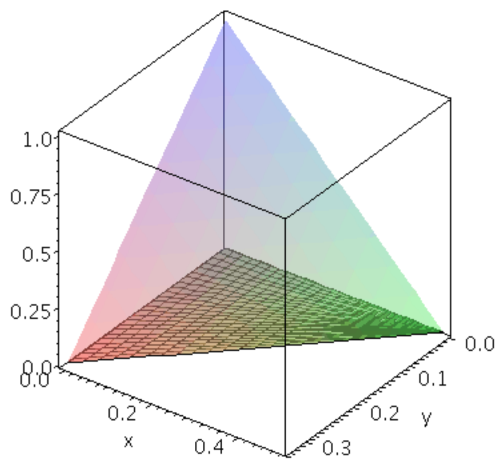
Thus

$$C = \frac{e}{\pi(e-1)}$$

Problem 4. (20pts) Assume that $\delta = \sin(xyz)$ is the density of an object D whose total mass is given by the integral

$$M = \int_0^{1/2} \int_0^{\frac{1}{3}-\frac{2}{3}x} \int_0^{1-2x-3y} sin(xyz)\ dzdydx.$$

(a) Sketch the region D occupied by the object in xyz-space.



(b) Set up the interated integral in the order $\int\int\int \sin(xyz)\ dydzdx$.

Soln : $$M = \int_0^{1/2} \int_0^{1-2x} \int_0^{1/3-2/3x-1/3z} \sin(xyz)\ dydzdx.$$

[DO NOT SOLVE]

(c) Give an integral formula for the center of mass $(\bar{x}, \bar{y}, \bar{z})$.

Soln :
$$\begin{aligned}
\bar{x} &= \frac{1}{M} \int_0^{1/2} \int_0^{1-2x} \int_0^{1/3-2/3x-1/3z} x \sin(xyz)\ dydzdx \\
\bar{y} &= \frac{1}{M} \int_0^{1/2} \int_0^{1-2x} \int_0^{1/3-2/3x-1/3z} y \sin(xyz)\ dydzdx \\
\bar{z} &= \frac{1}{M} \int_0^{1/2} \int_0^{1-2x} \int_0^{1/3-2/3x-1/3z} z \sin(xyz)\ dydzdx.
\end{aligned}$$

[DO NOT SOLVE]

Problem 5. (20pts) Consider the map (t, x) to $(\bar{t}, \bar{x})$ given by

$$t = \bar{t}\cosh\theta + \bar{x}\sinh\theta, \qquad (1)$$
$$x = \bar{t}\sinh\theta + \bar{x}\cosh\theta \qquad (2)$$

where $cosh\,\theta = \frac{e^{\theta}+e^{-\theta}}{2}$, $sinh\,\theta = \frac{e^{\theta}-e^{-\theta}}{2}$, and θ is a constant.

(a) Define what it means for the map (t, x) to $(\bar{t}, \bar{x})$ to be *area preserving*.

Soln: The map is *area preserving* if

$$\int\int_{\mathcal{R}_{tx}} dt dx = \int\int_{\mathcal{R}_{\bar{t}\bar{x}}} d\bar{t}d\bar{x}$$

for every two dimensional area $\mathcal{R}_{tx}$ in the (t, x)-plane that maps to $\mathcal{R}_{\bar{t}\bar{x}}$ in the $(\bar{t}, \bar{x})$-plane.

(b) Show that (1), (2) defines an area preserving map.

Soln: For every two dimensional area $\mathcal{R}_{tx}$ in the (t, x)-plane, we have

$$\int\int_{\mathcal{R}_{tx}} dt dx = \int\int_{\mathcal{R}_{\bar{t}\bar{x}}} |J(\bar{t}, \bar{x})|\ d\bar{t}d\bar{x}$$

where

$$|J(\bar{t}, \bar{x})| = \begin{pmatrix} \frac{\partial t}{\partial \bar{t}} & \frac{\partial t}{\partial \bar{x}} \\ \frac{\partial x}{\partial \bar{t}} & \frac{\partial x}{\partial \bar{t}} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} = \cosh^2\theta - \sinh^2\theta = 1.$$

Therefore,

$$\int\int_{\mathcal{R}_{tx}} dt dx = \int\int_{\mathcal{R}_{\bar{t}\bar{x}}} d\bar{t}d\bar{x},$$

and the map is area preserving.

(c) Is (1), (2) a *linear* or *nonlinear* map? **Soln:** LINEAR

(The map $(t, x) \leftrightarrow (\bar{t}, \bar{x})$ is a *Lorentz transformation*, the source of the "relativity of time" in Einstein's Theory of Relativity.)