Name: Solutions

Student ID#: ____________

Section: ____________

Midterm Exam 1

~~Wednesday April 25~~ → Friday, Feb 1, 2013

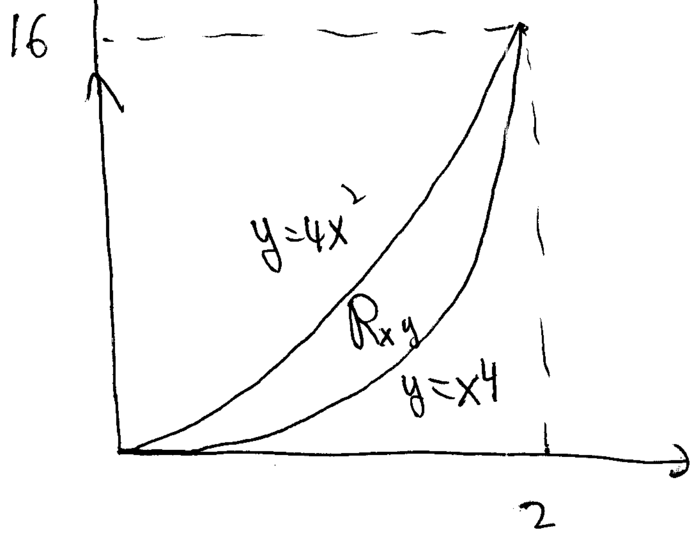
MAT 21D, Temple, Winter 2013

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): **(a)** Sketch the region of integration $\mathbf{R}_{xy}$ determined by the iterated integral

$$\int_0^2 \int_{x^4}^{4x^2} \tan(xy)\, dy dx. \tag{1}$$



$x^4 = 4x^2$

$x^2 = 4, 0$

$x = \pm 2, 0$

(b) Rewrite (1) with order of integration reversed. (Do not evaluate)

$$\int_0^{16} \int_{\frac{\sqrt{y}}{2}}^{\sqrt[4]{y}} \tan xy \, dx\, dy$$

$y = x^4 \Leftrightarrow x = \sqrt[4]{y}$

$y = 4x^2 \Leftrightarrow x = \frac{\sqrt{y}}{2}$

Problem #2 (20pts): A metal plate of density $1kg/m^2$ lies between the two curves $y = x^4$ and $y = 4x^2$. Find the KE stored in rotating the plate about the y-axis at $\omega = 2$ radians/sec. (Put in the units!)

$$KE = \frac{1}{2} I \omega^2 \qquad I = \int_0^2 \int_{x^4}^{4x^2} x^2 \, dy \, dx$$

$$= \int_0^2 x^2 \cdot y \Big]_{y=x^4}^{y=4x^2} dx = \int_0^2 x^2(4x^2 - x^4)\, dx$$

$$= \int_0^2 4x^4 - x^6 \, dx = 4\frac{x^5}{5} - \frac{x^7}{7}\Big]_0^2 = \frac{2^2 \cdot 2^5}{5} - \frac{2^7}{7}$$

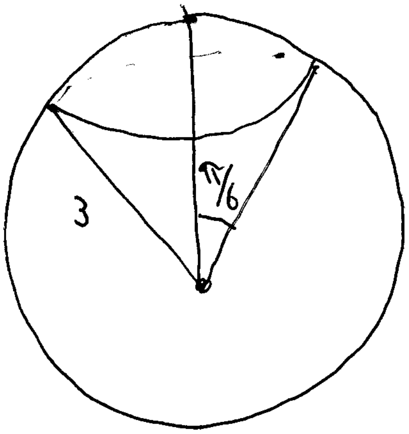
$$= 2^7\left(\frac{1}{5} - \frac{1}{7}\right) = 2^7\left(\frac{2}{35}\right) = \frac{2^8}{35}$$

$$\therefore \quad KE = \frac{1}{2}\left(\frac{2^8}{35}\right) \cdot 2^2 = \frac{2^9}{35} \underbrace{kg \frac{m^2}{s^2}}$$

$$[KE] = \frac{\text{Mass} \times \text{Length}^2}{\text{Time}^2}$$

Problem #3 (20pts): Integrate in spherical coordinates (ρ, ϕ, θ) to find the volume of the region obtained by removing the cone $\phi \le \pi/6$ from the sphere $x^2 + y^2 + z^2 = 9$.

$$V = \int_0^{2\pi} \int_{\pi/6}^{\pi} \int_0^{1 \nearrow 3} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$



Region: $0 \le 2\pi \le \theta$, $\cancel{\pi}\,\pi/6 \le \varphi \le \pi$

$0 \le \rho \le 3$

Rectangular ⇒ any order of integrat[ion] OK

$$= \int_0^{2\pi} \int_{\pi/6}^{\pi} \frac{\rho^3}{3}\Big]_0^{1 \nearrow 3} \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} \int_{\pi/6}^{\pi} \frac{1}{3} \nearrow 9 \, \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3} \nearrow 9 \int_0^{2\pi} -\cos\varphi \Big]_{\pi/6}^{\pi} d\theta = \frac{1}{3} \nearrow 9 \int_0^{2\pi} \Big[\underset{\overset{\uparrow}{\boxed{\sqrt{3}\; 1/2}}}{\cos \pi/6} - \underset{\overset{\uparrow}{-1}}{\cos \pi}\Big] d\theta$$

~~$= \frac{1}{3} \nearrow 27 \cdot \frac{3}{2} \, \theta \Big]_0^{2\pi} = \frac{1}{2} \nearrow 27 \cdot 2\pi = \pi \; 27$~~

Problem #4 (20pts): The change of variables formula for a general mapping $x = g(u, v)$, $y = h(u, v)$ is:

$$\int\int_{\mathcal{R}_{xy}} f(x, y)dxdy = \int\int_{\mathcal{R}_{uv}} f(g(u, v), h(u, v))J\, dudv.$$

Give the formula for the amplification factor J, and use this to derive the amplification factor $dxdy = rdrd\theta$ for the mapping to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \underbrace{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}_{\text{(not necessary)}}$$

For polar coordinates:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r \quad ✓$$

Problem #5 (20pts):
Use polar coordinates (and the trick in class) to evaluate $\int_{-\infty}^{+\infty} e^{-x^2}dx$, the area under the Gaussian.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_{-\infty}^{\infty} e^{-x^2}\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)dx$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = I^2.$$

But using polar coordinate,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta \qquad z=r^2 \quad dz=2r\,dr$$

$$= \int_0^{2\pi}\frac{1}{2}\int_0^{\infty} e^{-z}\,dz\,d\theta = \frac{1}{2}\cdot 2\pi\left[-e^{-z}\right]_{z=0}^{z=\infty}$$

$$= \pi\left[-0+1\right] = \pi. \quad \text{Thus } I^2 = \pi$$

$$I = \sqrt{\pi}$$ ✓