Name: Solutions

Student ID#: ____________

Section: ____________

Midterm Exam 2

Monday March 4

MAT 21D, Temple, Winter 2013

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): A particle of mass $m = \overset{2}{\cancel{3}}\ kg$ moves along a trajectory given by

$$\mathbf{r}(t) = \cos 2t\ \mathbf{i} + \sin 2t\ \mathbf{j} + t\ \mathbf{k},$$

where t is in seconds and $\mathbf{r}$ is in km. **(i)** At each time t find the following, give the correct dimensions, or say its dimensionless:

(a) The velocity vector $\mathbf{v}(t)$ $\quad \vec{v} = -2\sin 2t\ \underset{\sim}{i} + 2\cos 2t\ \underset{\sim}{j} + \underset{\sim}{k}$

(b) The speed $v(t)$ $\quad \|\vec{v}\| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 1} = \sqrt{4+1} = \sqrt{5}$

(c) The acceleration vector $\mathbf{a}(t)$ $\quad \vec{a} = \vec{r}''(t) = \vec{v}'(t) = -4\cos 2t\ \underset{\sim}{i} - 4\sin 2t\ \underset{\sim}{j}$

(d) The unit tangent vector $\mathbf{T}(t)$

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = -\frac{2}{\sqrt5}\sin 2t\ \underset{\sim}{i} + \frac{2}{\sqrt5}\cos 2t\ \underset{\sim}{j} + \frac{1}{\sqrt5}\underset{\sim}{k}$$

(e) The unit normal $\mathbf{N}(t)$

$$\vec{N} = \frac{d\vec{T}}{dt}\frac{1}{\left\|\frac{dT}{dt}\right\|} = \frac{\boxed{\sqrt5}}{2}\left(-\frac{4}{\sqrt5}\cos 2t\ \underset{\sim}{i} - \frac{4}{\sqrt5}\sin t\ \underset{\sim}{j}\right) = -\left(\cos 2t\ \underset{\sim}{i} + \sin 2t\ \underset{\sim}{j}\right)$$

(f) The curvature $\kappa(t)$.

$$K = \frac{1}{|v|}\left|\frac{dT}{dt}\right| = \frac{1}{\sqrt5}\left(\frac{4}{\sqrt5}\right) = \frac{4}{5}$$

(g) The length of the component of $\mathbf{a}(t)$ in direction of $\mathbf{T}$. $\quad a_T = 0$

Problem #2 (20pts): For the curve of Problem **1**:
(a) Find the arclength from $t = 0$ to $t = 1$.

$$\Delta s = \int_0^1 |\vec{v}|\, dt = \sqrt{5}\Big]_0^1 = \sqrt{5}$$

(b) Write down and evaluate the correct line integral for the work done by the $\mathbf{F} = x\mathbf{i} \quad y\mathbf{j}$ between $t = 0$ to $t = 1$. Include the correct units.

$$\int_C \vec{F}\cdot\vec{T}\, ds = \int_0^1 \vec{F}\cdot\vec{v}\, dt$$

$$= \int_0^1 \overrightarrow{\langle -\cos 2t, \sin 2t\rangle}\cdot\overrightarrow{\langle -2\sin 2t, +2\cos 2t\rangle}\, dt$$

$$= \int_0^1 2\cos 2t \sin 2t + 2\cos 2t \sin 2t\, dt$$

$u = \sin 2t$

$$= 4\int_0^1 \cos 2t \sin 2t\, dt$$

$du = 2\cos 2t$

$$= 2\int_{t=0}^{t=1} u\, du = 2\frac{u^2}{2}\Big]_{t=0}^{t=1} = \sin^2 2t\Big]_0^1 = \boxed{\sin^2 2}$$

Problem #3 (20pts): Let

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + (xz - y)\mathbf{j} + (xy + 1)\mathbf{k}.$$

Find:

(a) Div **F** $= \dfrac{\partial}{\partial x}\left(\cancel{yz}\right) + \dfrac{\partial}{\partial y}\left(xz - y\right) + \dfrac{\partial}{\partial z}\left(xy - \cancel{\tfrac{1}{2}y^2} + 1\right)$

$0 \qquad\qquad 0$

$= -1$

(b) Curl **F** $= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ yz & xz - y & xy - \frac{1}{2}y^2 + 1 \end{vmatrix}$

$= \underset{\sim}{i}\left((x - y) - (x - y)\right) - \underset{\sim}{j}\left(y - y\right) + \underset{\sim}{k}\left(z - z\right) = \underset{\sim}{0}$

(c) Find f such that $\mathbf{F} = \nabla f$

1 $\frac{\partial f}{\partial x} = yz \qquad f = \int_x yz = xyz + g(y,z)$

2 $\frac{\partial f}{\partial y} = \underbrace{xz - y = \frac{\partial}{\partial y}\left(xyz + g(y,z)\right) = xz + \frac{\partial g}{\partial y}}$

$\frac{\partial g}{\partial y} = -y \qquad g = \int_y -y = -\frac{1}{2}y^2 + h(z)$

3 $\frac{\partial f}{\partial z} = \underbrace{xy + 1 = \frac{\partial}{\partial z}\left(xyz - \frac{1}{2}y^2 + h(z)\right) = xy + h'(z)}$

$h'(z) = 1 \qquad h(z) = z + \text{Const}$

$$\boxed{f(x,y,z) = xyz - \frac{1}{2}y^2 + z + \text{Const}}$$

not needed

(d) Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ along any smooth curve C taking $A = (-1, 1, 2)$ to $B = (1, 1, -1)$.

$$\int_C \vec{F} \cdot \vec{T}\, ds = f(1,1,-1) - f(-1,1,2)$$

$$= 1 \cdot 1 \cdot (-1) - \tfrac{1}{2}(1)^2 + (-1)$$

$$-(-1)(1)(2) - \left(\tfrac{1}{2} 1^2\right) - 2$$

$$= -1 - \cancel{\tfrac{1}{2}} - 1$$

$$+\cancel{2} + \cancel{\tfrac{1}{2}} - \cancel{2}$$

$$= -2 \checkmark$$

Problem #4 (20pts): (a) Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field, where M, N, P are assumed to be given functions of (x, y, z). Use Leibniz's substitution principle to show the following are equal: (Here $\mathbf{r}(t)$ denotes any parameterization of curve C.)

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_C M dx + N dy + P dz.$$

$$\int_C F \cdot T\, ds = \int_C \vec{F} \cdot \underbrace{\vec{T} \frac{ds}{dt}}_{\vec{v}} dt = \int_C \vec{F} \cdot \underbrace{\vec{v}\, dt}_{d\vec{r}}$$

$$\frac{d\vec{r}}{dt} = \vec{v} \Rightarrow d\vec{r} = \vec{v}\, dt \Rightarrow \quad = \int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = \overrightarrow{(M, N, P)}$$

$$\vec{F} \cdot \vec{v} = \overrightarrow{(M, N, P)} \cdot \overrightarrow{\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)}$$

$$\vec{F} \cdot \vec{v}\, dt = \vec{F} \cdot \overrightarrow{(dx, dy, dt)} = M dx + N dy + P dz$$

$$\therefore \int_C \vec{F} \cdot \vec{v}\, dt = \int_C M dx + N dy + P dz$$

(b) Assume further that $F = \nabla f$ for some scalar function $f(x, y, z)$. Prove that

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = f(\mathbf{r}(b)) - f(\mathbf{r}(a)),$$

where $\mathbf{r}(t)$ is any smooth parameterization of C with $\mathbf{r}(a) = A$, $\mathbf{r}(b) = B$.

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F} \cdot \vec{v}\, dt \underset{\vec{F} = \nabla f}{=} \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

Problem #5 (20pts): Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be the trajectory of a planet moving in a plane with the Sun at the center $(x, y) = 0$. In the Newton-Kepler problem we showed that if $x = r\cos\theta$ and $y = r\sin\theta$, then differentiating $\mathbf{r}(t)$ twice and simplifying (using equal area in equal time) led to

$$\begin{aligned} \ddot{x}\cos\theta + \ddot{y}\sin\theta &= \ddot{r} - \frac{H^2}{r^3} \qquad (1) \\ -\ddot{x}\sin\theta + \ddot{y}\cos\theta &= 0. \qquad (2) \end{aligned}$$

Use these to solve for the acceleration vector $\mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$, and show that the acceleration vector points in the direction of the position vector $\mathbf{r}(t)$, (i.e., toward the sun).

Mult (1) by $\cos\theta$, & (2) by $\sin\theta$ & subtract:

$$\ddot{x} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\cos\theta = \frac{1}{r}\left(\ddot{r} - \frac{H^2}{r^3}\right)x$$

Mult (1) by $\sin\theta$, & (2) by $\cos\theta$ & add:

$$\ddot{y} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\sin\theta = \frac{1}{r}\left(\ddot{r} - \frac{H^2}{r^3}\right)y$$

$$\vec{a} = \overrightarrow{(\ddot{x}, \ddot{y})} = \underbrace{\frac{1}{r}\left(\ddot{r} - \frac{H^2}{r^3}\right)}_{\text{scalar}} \underbrace{\overrightarrow{(x, y)}}_{\vec{r}(t)}$$

"acceleration has direction of position vector $\vec{r}(t)$"