Name: Solutions

Student ID#: ______________

Section: ______________

Midterm Exam 2

Wednesday May 23

MAT 21D, Temple, Spring 2012

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): A particle of mass $m = 3\ kg$ moves along a trajectory given by

$$\mathbf{r}(t) = 2\cos t\ \mathbf{i} + 2\sin t\ \mathbf{j} + 3t\ \mathbf{k},$$

where t is in seconds and $\mathbf{r}$ is in km. **(i)** At each time t find the following, give the correct dimensions, or say its dimensionless:

(a) The velocity vector $\mathbf{v}(t)$ $= -2\sin t\ \underset{\sim}{i} + 2\cos t\ \underset{\sim}{j} + 3\underset{\sim}{k}\quad \frac{km}{s}$

(b) The speed $v(t)$ $= |\vec{v}(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 9} = \sqrt{13}\quad \frac{km}{s}$

(c) The acceleration vector $\mathbf{a}(t)$ $= \dot{\vec{v}}(t) = -2\cos t\ \underset{\sim}{i} - 2\sin t\ \underset{\sim}{j}\quad \frac{km}{s^2}$

(d) The unit tangent vector $\mathbf{T}(t)$ $= \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{13}}\left(-2\sin t\ \underset{\sim}{i} + 2\cos t\ \underset{\sim}{j} + 3\underset{\sim}{k}\right)$

dimensionless

(e) The unit normal $\mathbf{N}(t)$ $= \left|\frac{d\vec{T}}{dt}\right|^{-1}\frac{d\vec{T}}{dt} = \left|\frac{d\vec{T}}{dt}\right|^{-1}\frac{1}{\sqrt{13}}\left(-2\cos t\ \underset{\sim}{i} - 2\sin t\ \underset{\sim}{j}\right)$

$= -\cos t\ \underset{\sim}{i} - \sin t\ \underset{\sim}{j}$ dimensionless

(f) The curvature $\kappa(t)$. $= \left\|\frac{dT}{ds}\right\| = \left\|\frac{dT}{dt}\right\|\left|\frac{dt}{ds}\right| = \frac{2}{\sqrt{13}}\cdot\frac{1}{\sqrt{13}} = \frac{2}{13}\ \frac{1}{km}$

(g) The length of the component of $\mathbf{a}(t)$ in direction of $\mathbf{T}$.

$|\vec{a}\cdot\vec{T}| = \left(-2\cos t\ \underset{\sim}{i} - 2\sin t\ \underset{\sim}{j}\right)\cdot\left(-2\sin t\ \underset{\sim}{i} + 2\cos t\ \underset{\sim}{j} + 3\underset{\sim}{k}\right)\frac{1}{\sqrt{13}}$

$= 0\ \checkmark\quad \frac{km}{s^2}$

(ii) Find the arclength from $t = 0$ to $t = 1$.

$$\int_0^1 \frac{ds}{dt}\, dt = \int_0^1 \sqrt{13}\, dt = \sqrt{13}$$

(iii) Write down and evaluate the correct line integral for the work done by the Force creating the motion between $t = 0$ to $t = 1$. Include the correct units.

$$\Delta W = \int_0^1 \vec{F} \cdot \vec{v}\, dt = \int_0^1 m\, \vec{a} \cdot \vec{v}\, dt = \int_0^1 3\, \vec{a} \cdot \vec{v}\, dt$$

$$\vec{a} \cdot \vec{v} = \overrightarrow{-2(\cos t, \sin t)} \cdot (+2)\overrightarrow{\left(-\sin t, \cos t, \tfrac{3}{2}\right)} = 0$$

$$\Delta W = 0$$

Problem #2 (20pts): Let

$$\mathbf{F}(x, y, z) = 4xy\mathbf{i} + (2x^2 - 3z)\mathbf{j} + (-3y + 1)\mathbf{k}.$$

$$M \qquad N \qquad P$$

Find:

(a) Div **F** $= M_x + N_y + P_z$

$= 4y + 0 + 0 = 4y$

(b) Curl **F** $= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ 4xy & 2x^2-3z & -3y+1 \end{vmatrix}$

$= i\left(-3-(-3)\right) - \underset{\sim}{j}\left(0-0\right) + \underset{\sim}{k}\left(4x-4x\right) = 0$

(c) f such that $\mathbf{F} = \nabla f$

$$f = \int_x 4xy\,dx = 2x^2y + g(y,z)$$

$$\frac{\partial f}{\partial y} = 2x^2 + \frac{\partial g}{\partial y} = 2x^2 - 3z \Rightarrow \frac{\partial g}{\partial y} = -3z$$

$$g = -3zy + h(z)$$

$$\Rightarrow f = 2x^2y - 3zy + h(z)$$

$$\frac{\partial f}{\partial z} = -3y + h'(z) = -3y + 1 \Rightarrow h'(z) = 1 \quad h(z) = z$$

$$\Rightarrow f = 2x^2y - 3zy + z$$

Check: $\nabla f = \overrightarrow{(4xy, 2x^2 - 3z, -3y+1)} = \vec{F}$ ✓

(d) Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where C is the *hypo-gastro-geometric-meso-cycloid*, a smooth curve that takes $A=(1,-1,2)$ to $B=(-1,1,1)$.

$$\int_C \vec{F}\cdot T\,ds = f(B)-f(A) = f(-1,1,1)-f(1,-1,2)$$

$$= 2x^2y-3zy+z\Big]_{(+1,-1,2)}^{(-1,+1,1)} = -6$$

$$=\left(2\cdot 1^2(-1)-3\cdot 2(-1)+2\right)$$

$$-\left(2(-1)^2\cdot 1-3\cdot 1\cdot 1+1\right)$$

$$= -2+6+\cancel{2}-\cancel{2}+3+1 = 8$$

Problem #3 (20pts): (a) Use the Leibniz substitution principle to show that in general, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a parameterization of curve C for $a \le t \le b$, and $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field, then the following are equal:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_C M dx + N dy + P dz.$$

$$\int_C F \cdot T\, ds = \int_C F \cdot dr = \int_C \cancel{F} \cdot \vec{v}\, dt$$

$$\frac{d\vec{F}}{dt} = \vec{v}$$

$$d\vec{r} = \vec{v}\, dt = \vec{T}\, ds$$

$$\& \int_C \vec{F} \cdot d\vec{r} = \int_C \overrightarrow{(M,N,P)} \cdot \overrightarrow{(dx,dy,dz)}$$

$$d\vec{r} = \overrightarrow{(dx,dy,dz)}$$

$$= \int_C M dx + N dy + P dz$$

(b) Assume further that $F = \nabla f$ for some scalar function $f(x, y, z)$. Prove that

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = f(\overset{\vec{r}(b)}{\cancel{Q}}) - f(\overset{\vec{r}(a)}{\cancel{P}}).$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F} \cdot \vec{v}\, dt$$

$$\vec{F} \cdot \vec{v} = \vec{F}(x(t), y(t), z(t)) \cdot (\dot{x}(t), \dot{y}(t), \dot{z}(t))$$

$$= \frac{d}{dt} f(x(t), y(t), z(t))$$

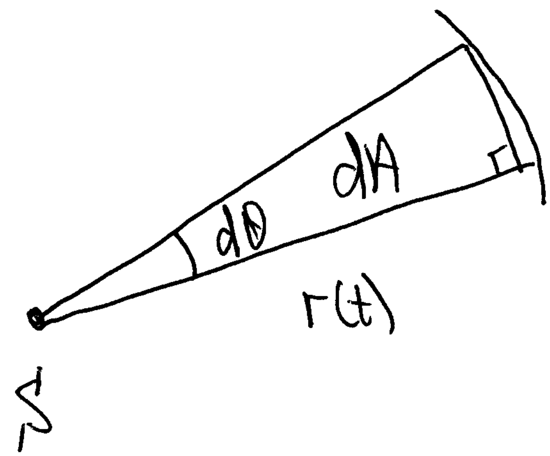
$$= \int_a^b \frac{d}{dt} f(x(t), y(t), z(t))\, dt = (\vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$$

✓

Problem #4 (20pts): Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be the trajectory of a planet moving in a plane with the Sun at the center $(x, y) = 0$.

(a) Write $\mathbf{r}(t)$ in polar coordinates (r, θ) and derive the condition that the planet sweeps out equal area in equal time.

$x(t) = r(t)\cos\theta(t) \qquad y(t) = r(t)\sin\theta(t)$



$$dA = \frac{1}{2} r(t) \cdot r(t)\, d\theta = \frac{1}{2} r(t)^2 \dot{\theta}\, dt$$

$$\frac{dA}{dt} = \frac{1}{2} r(t)^2 \dot{\theta} = H = \text{const} \qquad (*)$$

(b) Use Part (a) to show that if the planet sweeps out equal area in equal time, then $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$.

Diff (*): $0 = \frac{1}{2}\cdot 2r\dot{r}\dot{\theta} + \frac{1}{2}r^2\ddot{\theta}$

$$0 = r\dot{r}\dot{\theta} + \frac{1}{2}r^2\ddot{\theta}$$

$$\therefore \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{2}{r}\left\{ r\dot{r}\dot{\theta} + \frac{1}{2}r^2\ddot{\theta} \right\} = 0$$

Problem #5 (20pts): Recall that the Curl form of Green's Theorem is

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int\!\!\int_R N_x - M_y\,dA,$$

and the Div form of Green's Theorem is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int\!\!\int_R M_x + N_y\,dA,$$

so named because when $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then $Curl\,\mathbf{F} = (N_x - M_y)\mathbf{k}$ and $Div\,\mathbf{F} = M_x + N_y$. Assume the Curl form of Green's theorem and derive the Div form from the Curl form.

Assume: Curl form for $G = (\bar{M}, \bar{N})$

$$\int_C \vec{G}\cdot\vec{T}\,ds = \iint_R \bar{N}_x - \bar{M}_y\,dxdy \qquad (*)$$

set $M = \bar{N}$, $N = -\bar{M}$

so RHS $= \iint_R \operatorname{div}\vec{F}\,dxdy$

with $\vec{F} = (M, N)$

Then: $\vec{G}\cdot\vec{T} = (\bar{M},\bar{N})\cdot(T_x, T_y) = \underbrace{(\bar{N}, -\bar{M})}_{\vec{F}}\cdot\underbrace{(T_y, -T_x)}_{\vec{n}}$

$= \vec{F}\cdot\vec{n}$

Conclude: LHS $(*)$ $= \int_C \vec{F}\cdot\vec{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dxdy =$ RHS $(*)$