Name: ____________________

Student ID#: ____________________

Section: ____________________

Midterm Exam 2–Solutions

Wednesday, May 18

MAT 21D, Romik/Temple, Spring 2016

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): (a) Use integration in spherical coordinates to calculate the volume of a ball of radius R.

Solution:

$$\begin{aligned}
Vol(B_R) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi \; d\rho d\phi d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi \, d\phi\right)\left(\int_0^{R} \rho^2 d\rho\right) \\
&= 2\pi \times 2 \times \frac{R^3}{3} = \frac{4\pi R^3}{3}.
\end{aligned}$$

(b) Use integration in spherical coordinates to find the moment of inertia of a ball of radius R and constant density δ_0 about the z-axis. (Recall $I_z = \int\int\int_D r^2 \delta dV$.)

Solution:

$$\begin{aligned}
Vol(B_R) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\rho^2\sin\phi\ \delta_0\ d\rho d\phi d\theta \\
&= \delta_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (\rho\sin\phi)^2\rho^2\sin\phi\ \delta_0\ d\rho d\phi d\theta \\
&= \delta_0 \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin^3\phi\, d\phi\right)\left(\int_0^{R}\rho^4 d\rho\right) \\
&= \delta_0\,(2\pi)\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Bigg|_{\phi=0}^{\phi=\pi}\left(\frac{R^5}{5}\right) \\
&= \delta_0 2\pi \times \left(2 - \frac{2}{3}\right) \times \frac{R^5}{5} = \frac{8\pi R^5}{15}.
\end{aligned}$$

Problem #2 (20pts): A particle moves along a trajectory given by

$$\mathbf{r}(t) = R\sin\omega t\ \mathbf{i} + R\cos\omega t\ \mathbf{j} + R\,t\ \mathbf{k}.$$

(a) Find the velocity vector $\mathbf{v}(t)$ and the speed $v = \frac{ds}{dt}$.

$$\mathbf{v}(t) = R\omega\cos\omega t\ \mathbf{i} - R\omega\sin\omega t\ \mathbf{j} + R\ \mathbf{k}.$$

$$v(t) = R\sqrt{\omega^2+1}.$$

(b) Express ω in terms of R and v.

$$\omega = \sqrt{\left(\frac{v}{R}\right)^2 - 1}$$

(c) Find the acceleration vector $\mathbf{a}(t)$

$$\mathbf{a}(t) = -R\omega^2\sin\omega t\ \mathbf{i} - R\omega^2\cos\omega t\ \mathbf{j}.$$

(d) Find the unit tangent vector $\mathbf{T}(t)$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{\omega}{\sqrt{\omega^2+1}} \cos \omega t \ \mathbf{i} - \frac{\omega}{\sqrt{\omega^2+1}} \sin \omega t \ \mathbf{j} + \frac{1}{\sqrt{\omega^2+1}} \ \mathbf{k}.$$

(e) Find the length of the component of $\mathbf{a}(t)$ in direction of $\mathbf{T}$.

$$\mathbf{a}\cdot\mathbf{T} = \overrightarrow{\left(-R\omega^2 \sin \omega t, -R\omega^2 \cos \omega t\right)} \cdot \overrightarrow{\left(\frac{\omega}{\sqrt{\omega^2+1}} \cos \omega t, -\frac{\omega}{\sqrt{\omega^2+1}} \sin \omega t, \frac{1}{\sqrt{\omega^2+1}}\right)} = 0.$$

Problem #3 (20pts): A projectile is fired at a velocity of $30m/s$ from a cannon placed at the origin at an angle α degrees relative to the x-axis. Assuming a uniform gravitational acceleration of $g \approx -10m/s^2$ in the negative y direction is the only force acting, derive a formula for the maximal height in terms of α.

Solution: $y''(t) = -10$, so $y'(t) = -10t + v_0 = -10t + 30\sin(\alpha)$. Maximum height is the value of y when $y'(t) = 0$. Thus $t_{max} = 3\sin\alpha$, and so

$$y(t_{max}) = -5(t_{max})^2 + 30\sin(\alpha)t_{max} = -45\sin^2\alpha + 90\sin^2(\alpha) = 45\sin^2(\alpha)\ meters.$$

Problem #4 (20pts): (a) Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field, where M, N, P are assumed to be given functions of (x, y, z). Use Leibniz's substitution principle to show the following are equal: (Here $\mathbf{r}(t)$ denotes any parameterization of curve C.)

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_C M dx + N dy + P dz = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \mathbf{F} \cdot \mathbf{v}\ dt.$$

Solution: Given $\mathbf{r}(t)$ we have

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \frac{ds}{dt}\mathbf{T}.$$

Thus

$$\mathbf{T}ds = \mathbf{v}dt = d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}.$$

So

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_C \mathbf{F} \cdot \mathbf{v}\ dt = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_C M dx + N dy + P dz.$$

(b) Assume further that a mass m is moving according to $\mathbf{F} = m\mathbf{a}$, where $F = \overrightarrow{(M, N, P)}$ is given force field. Assume the resulting motion moves m along a curve C between points A and B. Prove that the work done by F between A and B is equal to the change in kinetic energy. That is, show that the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is equal to $\frac{1}{2}mv_B^2 - \frac{1}{2}mv_B^2$. (Hint: Assume C is parametrized by $r(t)$ and use $\mathbf{F} = m\mathbf{a}$.)

Solution: Given $\mathbf{r}(t)$ with $m\mathbf{r}''(t) = \mathbf{F}$, we have

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\ ds &= \int_C m\mathbf{v}'(t) \cdot \mathbf{v}(t)\ dt = \int_C m\frac{1}{2}\frac{d}{dt}\left(\mathbf{v}(t) \cdot \mathbf{v}(t)\right)\ dt \\ &= \frac{1}{2}mv^2|_{t_A}^{t_B} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_B^2.\end{aligned}$$

(c) Assume further that $\mathbf{F} = \nabla f$, and C is any curve taking A to B. Prove that $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = f(B) - f(A)$. (Hint: Let $\mathbf{r}(t)$ parametrize C from $r(t_A) = A$ to $r(t_B) = B$.)

Solution:

$$\int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_{t_A}^{t_B} \nabla f \cdot \mathbf{v}\,dt = \int_{t_A}^{t_B} \frac{d}{dt} f(\mathbf{r}(t)\,dt = f(B) - f(A).$$

Problem #5 (20pts): Consider the following vector field in $\mathcal{R}^3$.

$$\mathbf{F}(x,y,z) = \{2xy - e^z \sin x\}\,\mathbf{i} + x^2\mathbf{j} + e^z \cos x\mathbf{k}.$$

(a)(10pts) Show $\mathbf{F}$ is irrotational ($Curl\mathbf{F} = 0$), and hence conservative, (because $\mathcal{R}^3$ is simply connected).

Solution:

$$\begin{aligned}
Curl\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - e^z \sin x & x^2 & e^z \cos x \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y} e^z \cos x - \frac{\partial}{\partial z} x^2\right) - \mathbf{j}\left(\frac{\partial}{\partial x} e^z \cos x - \frac{\partial}{\partial z}(2xy - e^z \sin x)\right) \\
&\qquad + \mathbf{k}\left(\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y}(2xy - e^z \sin x)\right) \\
&= 0.
\end{aligned}$$

(b)(10pts) Derive a scalar function f such that $\mathbf{F} = \nabla f$.

Solution: $\frac{\partial f}{\partial x} = \int_x 2xy - e^z \sin x \ dx$ so

$$f = x^2 y + e^z \cos(x) + g(y,z);$$
$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2$$

so $g = g(z)$. Thus

$$\frac{\partial f}{\partial z} = e^z \cos x + g'(z) = e^z \cos x$$

so $g = 0$ and we have

$$f(x,y,z) = x^2 y + e^z \cos(x).$$

(c)(5pts) Compute $\int_C \mathbf{F} \cdot \mathbf{T} \ ds$ along the straight line taking $A = (0,0,0)$ to $B = (1,1,-1)$.

Solution:

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T} \ ds &= f(1,1,-1) - f(0,0,0) \\ &= \left\{1^2 \cdot 1 + e^{-1} \cos(1) - 0^2 \cdot 0 + e^0 \cos(0)\right\} = 2 + e \cos(1) - e\end{aligned}$$