Midterm 2 Solns

#1

(a) $\vec{v} = 2\cos t\, \underset{\sim}{i} + 2\sin t\, \underset{\sim}{j} + \underset{\sim}{k} = \vec{r}\,'(t)$

(b) $v = \|\vec{v}\| = \sqrt{4\cos^2 t + 4\sin^2 t + 1} = \sqrt{5}$

(c) $\vec{a} = \vec{r}\,''(t) = \vec{v}\,'(t) = -2\sin t\, \underset{\sim}{i} + 2\cos t\, \underset{\sim}{j}$

(d) $T = \dfrac{\vec{v}}{v} = \dfrac{2\cos t}{\sqrt{5}}\, \underset{\sim}{i} + \dfrac{2\sin t}{\sqrt{5}}\, \underset{\sim}{j} + \dfrac{1}{\sqrt{5}}\, \underset{\sim}{k}$

(e) $\vec{F} = m\vec{a} = 3(\vec{a}) = -6\sin t\, \underset{\sim}{i} + 6\cos t\, \underset{\sim}{j}$

(f) $\dfrac{d\vec{T}}{dt} \perp \vec{T}$ & $\dfrac{d\vec{T}}{dt} = -\dfrac{2\sin t}{\sqrt{5}}\, \underset{\sim}{i} + \dfrac{2\cos t}{\sqrt{5}}\, \underset{\sim}{j}$

$$\vec{N} = \frac{T'(t)}{|T'(t)|} = \frac{T'(t)}{\sqrt{\frac{4}{5}\sin^2 t + \frac{4}{5}\cos^2 t}} = \frac{T'(t)}{\sqrt{\frac{4}{5}}}$$

$$= \frac{\cancel{\sqrt{5}}}{\cancel{2}}\left(-\frac{\cancel{2}\sin t}{\cancel{\sqrt{5}}}\, \underset{\sim}{i} + \frac{\cancel{2}\cos t}{\cancel{\sqrt{5}}}\, \underset{\sim}{j}\right) = -\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j}$$

(g) $\text{arclength} = \displaystyle\int_2^3 v\, dt = \int_2^3 \sqrt{5}\, dt = \sqrt{5}(3-2) = \sqrt{5}$

#2

ⓐ $\text{Div}\vec{F} = M_x + N_y + P_z = \frac{\partial}{\partial x}(2xy\sin z) + \frac{\partial}{\partial y}(x^2\sin z) + \frac{\partial}{\partial z}(x^2y\cos z)$

$= 2y\sin z + 0 - x^2y\sin z = (2-x^2)y\sin z$

ⓑ $\text{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy\sin z & x^2\sin z & x^2y\cos z \end{vmatrix}$

$= \underset{\sim}{i}\left(x^2\cos z - x^2\cos z\right) - \underset{\sim}{i}\left(2xy\cos z - 2xy\cos z\right) + \underset{\sim}{k}\left(2x\sin z - 2x\sin z\right)$

$= 0$

ⓒ $\nabla f = \vec{F}$

ⓓ $\int_C \vec{F}\cdot T\,ds = f(B) - f(A) = (x^2y\sin z)(-2, 1, \pi/3) - (x^2y\sin z)(1, -1, \pi/4)$

$= (-2)^2\cdot 1\cdot \sin\pi/3 - 1^2(-1)\sin\pi/4$

$= 4\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = 2\sqrt{3} - \frac{1}{2}\sqrt{2}$

#3

(a) $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$

$\vec{r}(t) = \overrightarrow{(\cos\theta, \sin\theta, 1)}$ $d\vec{r} = \overrightarrow{(-\sin\theta, \cos\theta, 0)}$

$$= \int_0^{2\pi} \overrightarrow{\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z\right)} \cdot \overrightarrow{(-\sin\theta, \cos\theta, 0)}\, d\theta$$

$$= \int_0^{2\pi} \overrightarrow{\left(\frac{-\sin\theta}{\cos^2\theta+\sin^2\theta}, \frac{\cos\theta}{\cos^2\theta+\sin^2\theta}, 1\right)} \cdot \overrightarrow{(-\sin\theta, \cos\theta, 0)}\, d\theta$$

$$= \int_0^{2\pi} \sin^2\theta + \cos^2\theta\, d\theta = \boxed{2\pi}$$

(b) The singularity at $x=0$, $y=0$, $z\in\mathbb{R}$ makes the domain where $\vec{F}$ is defined not simply connected. (Loop around z-axis cannot be continuously contracted to a pt.)

#4 Rate at which mass leaves the box is

Flux thru ∂ = $\int_e \vec{F} \cdot \vec{n}\, ds$

"boundary"

where $\vec{F}$ = mass flux vector = $\delta \vec{v}$

$$= x\,\overrightarrow{(x, -y)} = \overrightarrow{(x^2, -xy)}$$

$$\therefore \int_e \vec{F} \cdot \vec{n}\, ds = \int_e (x^2, -xy) \cdot \vec{n}\, ds$$

$$= \int_0^1 \int_1^2 (x^2)_x + (-xy)_y \, dx\, dy$$

$$= \int_0^1 \int_1^2 2x - x \, dx\, dy$$

$$= 1 \cdot \int_1^2 x\, dx = \left.\frac{x^2}{2}\right]_1^2 = \frac{4}{2} - \frac{1}{2} = \boxed{\frac{3}{2}}$$

$\therefore$ Flux into box = $\boxed{-\frac{3}{2}\ \frac{kg}{s}}$

#5 a

$$\int_C \vec{F}\cdot\vec{T}\,ds \underset{\substack{\uparrow \\ ds = v\,dt}}{=} \int_C \vec{F}\cdot\underbrace{\vec{T}\cdot v}_{\vec{v}}\,dt = \int_C \vec{F}\cdot\underbrace{\vec{v}\,dt}_{d\vec{r}}$$

$$= \int_C \vec{F}\cdot d\vec{r}.$$

Also $\int_C M\,dx + N\,dy + P\,dz = \int_C M x'\,dt + N y'\,dt + P z'\,dt$

$$= \int_C \underbrace{\overrightarrow{(M,N,P)}}_{\vec{F}}\,\underbrace{\overrightarrow{(x',y',z')}}_{\vec{v}}\,dt = \int_C \vec{F}\cdot\vec{v}\,dt$$

b