Name: Solutions

Student ID#: __________

Section: __________

Midterm Exam 2

Wednesday May 18

MAT 21D, Temple, Spring 2011

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): A particle of mass $m = 2$ moves along a trajectory given by

$$\mathbf{r}(t) = 3\sin t\ \mathbf{i} + 3\cos t\ \mathbf{j} + t\ \mathbf{k}.$$

At each time t find:

(a) The velocity vector $\mathbf{v}$

$$\vec{V}(t) = \dot{\vec{r}}(t) = 3\cos t\, \underset{\sim}{i} - 3\sin t\, \underset{\sim}{j} + \underset{\sim}{k}$$

(b) The speed v

$$v(t) = \|\vec{v}(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 1^2} = \sqrt{10}$$

(c) The acceleration vector $\mathbf{a}$

$$\vec{a}(t) = \dot{\vec{V}}(t) = -3\sin t\, \underset{\sim}{i} - 3\cos t\, \underset{\sim}{j}$$

(d) The unit tangent vector $\mathbf{T}$

$$T(t) = \frac{1}{\|\vec{v}(t)\|} \vec{v}(t) = \frac{1}{\sqrt{10}} \{3\cos t\, \underset{\sim}{i} - 3\sin t\, j + k\}$$

(e) The force $\mathbf{F}$ on m

$$\vec{F} = ma = -3m(\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j}) \qquad m = 2$$

(f) The unit normal $\mathbf{N}$

$$\vec{N} = \frac{dT}{dt} \cdot \frac{1}{\|\frac{dT}{dt}\|} = \frac{3}{\sqrt{10}} (-\sin t\, \underset{\sim}{i} - \cos t\, \underset{\sim}{j}) \frac{\sqrt{10}}{3} = -(\sin t\, \underset{\sim}{i} + \cos t\, j)$$

(g) The curvature κ.

$$\kappa = \left\|\frac{dT}{ds}\right\| = \left\|\frac{dT}{dt}\frac{dt}{ds}\right\| = \frac{1}{v}\left\|\frac{dT}{dt}\right\| = \frac{1}{\sqrt{10}} \cdot \frac{3}{\sqrt{10}} = \frac{3}{10}$$

Problem #2 (20pts): Let C be the curve of Problem #1 defined for t between $t_a = 0$ and $t_b = \pi$:

$$\mathbf{r}(t) = 3\sin t\ \mathbf{i} + 3\cos t\ \mathbf{j} + t\ \mathbf{k}$$

(a) Find the arclength of C.

$$ds = \|\vec{v}\|dt = \sqrt{10}\,dt$$

$$S = \int_0^{\pi} \sqrt{10}\,dt = \sqrt{10}\,\pi$$

(b) Evaluate the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ assuming $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{\pi} \vec{F}\cdot\vec{v}\,dt = \int_0^{\pi} \overrightarrow{(x,y,z)}\,\overrightarrow{(3\cos t, -3\sin t, 1)}\,dt$$

$$= \int_0^{\pi} 3\cancel{\sin t\cos t} - 3\cancel{\cos t\sin t} + t\,dt = \frac{t^2}{2}\Big]_0^{\pi} = \frac{\pi^2}{2}$$

$\underline{\underline{\text{OR}}}$: $\vec{F} = \nabla\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2\right) = \nabla f$

$$\int_C \vec{F}\cdot\vec{T}\,ds = f(0,-3,\pi) - f(0,3,0)$$

$$= \cancel{\tfrac{1}{2}(-3)^2} + \frac{1}{2}\pi^2 - \cancel{\tfrac{1}{2}(3)^2} = \frac{1}{2}\pi^2 \checkmark$$

Problem #3 (20pts): Use the Leibniz substitution principle to show that in general, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a parameterization of curve C for $a \leq t \leq b$, and $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field, then the following are equal:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_C M dx + N dy + P dz.$$

$$T = \frac{\vec{v}}{\|\vec{v}\|}, \quad \|\vec{v}\| = \frac{ds}{dt} \quad \text{so} \quad T = \vec{v}\,\frac{dt}{ds}$$

$$\text{or} \quad T\,ds = \vec{v}\,dt \Rightarrow \int_C F \cdot T\,ds = \int_C F \cdot \vec{v}\,dt$$

$$\frac{d\vec{r}}{dt} = \vec{v} \quad \text{so} \quad d\vec{r} = \vec{v}\,dt \Rightarrow \int_C \vec{F} \cdot \vec{v}\,dt = \int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = \overrightarrow{(M, N, P)} \qquad \& \quad \vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} + z(t)\,\underset{\sim}{k}$$

$$\vec{F} \cdot \vec{v} = \overrightarrow{(M, N, P)} \cdot d\vec{r} = \overrightarrow{(M, N, P)} \cdot \overrightarrow{(dx, dy, dz)}$$

$$= M dx + N dy + P dz$$

(or any equivalent use of Liebniz notation)

Problem #4 (20pts): Assume that

$$\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$$

is a conservative force field so that $F = \nabla f$ for some scalar function $f(x,y,z)$. Assume that a mass m moves along a curve $\mathbf{r}(t)$ starting at point $P = \mathbf{r}(t_a)$ to point $Q = \mathbf{r}(t_b)$, $t_a \leq t \leq t_b$, and assume $\mathbf{F} = m\mathbf{a}$ at each t, so that the work done is $\Delta W = \int_P^Q \mathbf{F} \cdot \mathbf{T}\, ds$.

(a) Prove that $\Delta W = \frac{1}{2}m\|\mathbf{v}(t_b)\|^2 - \frac{1}{2}\mathbf{m}\|\mathbf{v}(t_a)\|^2$ where $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$. (That is, the work done is the change in Kinetic Energy.)

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_{t_a}^{t_b} \vec{F} \cdot \vec{v}\, dt = \int_{t_a}^{t_b} m\vec{a} \cdot \vec{v}\, dt$$

$$= \int_{t_a}^{t_b} \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v})\, dt = \frac{1}{2}m\|V_b\|^2 - \frac{1}{2}m\|V_a\|^2$$

(b) Prove that $\Delta W = f(Q) - f(P)$.

$$\int_{e} \vec{F} \cdot \vec{T}\, ds = \int_{t_a}^{t_b} \vec{F} \cdot \vec{v}\, dt = \int_{t_a}^{t_b} \nabla f \cdot \vec{v}\, dt$$

$$= \int_{t_a}^{t_b} \frac{d}{dt} f(\vec{r}(t))\, dt = f(\vec{r}(t_b)) - f(\vec{r}(t_a))$$

Problem #5 (20pts): Let C denote the unit circle in the (x, y)-plane oriented counterclockwise, and let $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$. $= M\underset{\sim}{i} + N\underset{\sim}{j}$

(a) Verify directly the *line integral* form of Green's Theorem:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_R N_x - M_y \, dA$$

$$\int_C \vec{F} \cdot T\, ds \underset{\uparrow}{=} \int_0^{2\pi} \vec{F} \cdot \vec{v}\, dt = \int_0^{2\pi} \overrightarrow{(y, -x)}\, \overrightarrow{(-\sin t, \cos t)}\, dt$$

$$\vec{r}(t) = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j} \qquad = \int_0^{2\pi} (\sin t, -\cos t)(-\sin t, \cos t)\, dt$$

$$\vec{v}(t) = -\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j}$$

$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt$$

$$= -2\pi$$

$$\iint_R N_x - M_y\, dx\,dy = \iint_{\mathbb{R}} (-1-1)\, dx\,dy = -2\pi \checkmark$$

(b) Evaluate the integrals on both sides and thereby verify directly the *flux* form of Green's Theorem:

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int\int_R M_x + N_y\,dA.$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} \overrightarrow{(y,-x)}\cdot\overrightarrow{(\cos t,\sin t)}\,\frac{ds}{dt}\,dt$$

$\vec{n}$ outer normal to circle $\vec{n}=\overrightarrow{(\cos t,\sin t)}$

$$= \int_0^{2\pi} \underbrace{\overrightarrow{(\sin t,-\cos t)}\cdot\overrightarrow{(\cos t,\sin t)}}_{0}\,\frac{ds}{dt}\,dt$$

$$=0$$

$$\iint_R M_x+N_y\,dA = \iint_R \frac{\partial}{\partial x}y+\frac{\partial}{\partial y}(-x)\,dA$$

$$=\iint_R 0\,dA = 0 \checkmark$$