Name: ______________________

Student ID#: ______________________

Section: ______________________

Midterm Exam 2

Wednesday, May 7

Temple, Winter 2018

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): A particle moves along a trajectory given by

$$\vec{\mathbf{r}}(t) = a\sin\omega t\ \mathbf{i} + a\cos\omega t\ \mathbf{j} + a\,t\ \mathbf{k}.$$

(a) Find a formula for the velocity vector $\vec{\mathbf{v}}(t)$, the speed $v = \frac{ds}{dt}$, and the unit tangent vector $\vec{\mathbf{T}}$.

Solution:

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = \omega a\cos\omega t\ \mathbf{i} - \omega a\sin\omega t\ \mathbf{j} + a\ \mathbf{k},$$

so

$$\frac{ds}{dt} = \|\vec{\mathbf{v}}\| = \sqrt{\omega^2 a^2\cos^2\omega t + \omega^2 a^2\sin^2\omega t + a^2} = a\sqrt{1+\omega^2}.$$

Thus

$$\vec{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{\omega\cos\omega t\ \mathbf{i} - \omega\sin\omega t\ \mathbf{j} +\ \mathbf{k}}{\sqrt{1+\omega^2}}$$

(b) Find a formula for the acceleration vector $\vec{\mathbf{a}}(t)$, and find tangential and normal components of the acceleration a_T, a_N such that $\vec{\mathbf{a}}(t) = a_T\vec{\mathbf{T}} + a_N\vec{\mathbf{N}}$.

Solution:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}''(t) = -\omega^2 a\sin\omega t\ \mathbf{i} - \omega^2 a\cos\omega t.$$

Since clearly $\vec{\mathbf{a}}(t)\cdot\vec{\mathbf{T}} = 0$, it follows that $\vec{\mathbf{a}}(t) = 0\cdot\vec{\mathbf{T}} + a_N\vec{\mathbf{N}}$, so

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}''(t) = \omega^2 a\left\{-\sin\omega t\ \mathbf{i} - \cos\omega t\ \mathbf{j}\right\} = \omega^2 a\vec{\mathbf{N}},$$

because $\vec{\mathbf{a}}$ is a positive multiple of $\vec{\mathbf{N}}$. Thus $a_T = 0$, $a_N = \omega^2 a$.

Problem #2 (20pts): Consider the following vector field in $\mathcal{R}^3$.

$$\vec{\mathbf{F}}(x,y,z) = \left\{4x^3y^3\right\}\mathbf{i} + (3x^4y^2 - 2yz^3)\mathbf{j} + (-3y^2z^2 + 2)\mathbf{k}.$$

Solution:

$$\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} = \frac{\partial}{\partial y}(-3y^2z^2 + 2) - \frac{\partial}{\partial z}(3x^4y^2 - 2yz^3) = 0$$

$$-\left\{\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right\} = -\left\{\frac{\partial}{\partial x}(-3y^2z^2 + 2) - \frac{\partial}{\partial z}(4x^3y^3)\right\} = 0$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}(3x^4y^2 - 2yz^3) - \frac{\partial}{\partial y}(4x^3y^3) = 0$$

(a) Show $\vec{\mathbf{F}}$ is irrotational (satisfies Curl $\vec{\mathbf{F}} = 0$), and hence conservative, (because $\mathcal{R}^3$ is simply connected).

(b) Find a scalar function f such that $\vec{\mathbf{F}} = \nabla f$.

(c) Compute $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ along the *hypo-centrific-unimodular-cantankeron*, an extremely complicated smooth curve that takes $A = (0, 0, 0)$ to $B = (1, -1, 1)$.

Problem #3 (20pts): (a) Let $r = \sqrt{x^2 + y^2}$. Calculate $\int_C \mathbf{F} \cdot \mathbf{T} ds$ directly, where C is the unit circle oriented counterclockwise, and

$$\vec{\mathbf{F}} = \frac{-y}{r^2}\mathbf{i} + \frac{x}{r^2}\mathbf{j}.$$

Solution: Let $\vec{\mathbf{r}}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$, and integrate as follows:

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds &= \int_0^{2\pi} \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}\, dt = \int_0^{2\pi} (-y, x) \cdot (-\sin t, \cos t)\, dt \\ &= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} dt = 2\pi. \end{aligned}$$

(b) Calculate Curl $\vec{\mathbf{F}}$.

Solution: Using $r_x = \frac{x}{r}$, etc, calculate

$$\mathrm{Curl}\,\vec{\mathbf{F}} = \{N_x - M_y\}\,\mathbf{k} = \left\{\left(\frac{1}{r^2} - 2\frac{x^2}{r^4}\right) + \left(\frac{1}{r^2} - 2\frac{y^2}{r^4}\right)\right\} = 0$$

(c) Reconcile **(a)** and **(b)** with Green's Theorem.

Solution: Green's theorem reads: $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \int\int_A \mathrm{Curl}\,\vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\; dA$ for any positively oriented curve C in the plane surrounding area A. Now $\mathrm{Curl}\,\vec{\mathbf{F}} = 0$ appears to make the RHS zero, when the left hand side is 2π, but in fact, the RHS does not have $\mathrm{Curl}\,\vec{\mathbf{F}} = 0$ at $r = 0$, so we cannot conclude from Green's Theorem that the RHS is zero.

Problem #4 (20pts): (a) Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, let C be a smooth curve that takes A to B, and let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Use Leibniz's substitution principle to show the following are equal:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\; ds = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}\; dt = \int_C M dx + N dy + P dz.$$

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} dt = \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt} dt = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = (M, N, P) \cdot (dx, dy, dz) = M dx + N dy + P dz$$

(b) Assume further that $\mathbf{F} = m\mathbf{a}$, and $\vec{\mathbf{F}}$ is conservative, so $\vec{\mathbf{F}} = -\nabla P$. Derive the principle of conservation of energy

$$\left\{\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right\} + \{P(B) - P(A)\} = 0.$$

(Hint: Integrate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ two different ways.)

Let C be the trajectory of a curve taking A to B parameterized by $\vec{\mathbf{r}}(t)$, where t is the time so $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}''(t)$ is the acceleration, $a \leq t \leq b$. Then first

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = -\int_a^b \nabla\mathbf{P} \cdot \vec{\mathbf{v}}\, dt = -\int_a^b \frac{d}{dt} P(\vec{\mathbf{r}}(t))\, dt = -(P(B) - P(A)$$

Also

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = -\int_a^b m\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}\, dt = -m\int_a^b \frac{d}{dt}(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})\, dt = -\frac{1}{2}mv_B^2 - \frac{1}{2}mv_B^2$$

Thus

$$\left\{\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right\} + \{P(B) - P(A)\} = 0.$$

Problem #5 (20pts): Consider Kepler's Laws under the simplifying assumption that the planets move in circular orbits with the sun at the center, (not a terrible approximation). In this case, each planet moves around a circle with position vector

$$\vec{\mathbf{r}}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j}, \tag{1}$$

where R and ω are constants which depend on the planet, and t is the time. (Assuming $r = R$ constant in (1) makes the orbit circular, and $\omega = const$ implies equal area in equal time, guaranteeing Kepler's first two laws.) Assume Kepler's third law

$$\frac{T^2}{R^3} = K, \tag{2}$$

where T is the period of the planet's rotation and K is a constant independent of the planet. Use (1) and (2) to derive Newton's Law of gravity

$$\vec{\mathbf{a}} = -G\frac{\vec{\mathbf{r}}}{r^3},$$

for circular orbits (1), and determine the value of the constant G.

Solution: The period T of (1) satisfies $\omega T = 2\pi$, so $T = 2\pi/\omega$. Now differentiating (1) twice gives

$$\vec{\mathbf{a}} = \vec{\mathbf{r}}''(t) = -\omega^2\vec{\mathbf{r}},$$

and since $T^2 = 4\pi^2/\omega^2$, we can use (2) to get

$$\frac{4\pi^2}{\omega^2R^3} = K, \quad \text{so} \quad \omega^2 = \frac{4\pi^2}{KR^3}.$$

Putting this into the above formula for $\vec{\mathbf{a}}$ gives

$$\vec{\mathbf{a}} = -\frac{4\pi^2}{K}\frac{\vec{\mathbf{r}}}{R^3},$$

which, since $r = R$ on circular orbits, gives Newton's law of gravity with $G = \frac{4\pi^2}{K}$.

(Continued...)

Extra Credit (2pts): Albert Einstein said J. Willard Gibbs was the greatest American Physicist of all time. Name one thing he did.

Solution: Gave the modern expression of vector calculus; Modern theory of Thermodynamics.