Name: Solutions

Student ID#: ______________

Section: ______________

Midterm Exam 1

Wednesday April 20

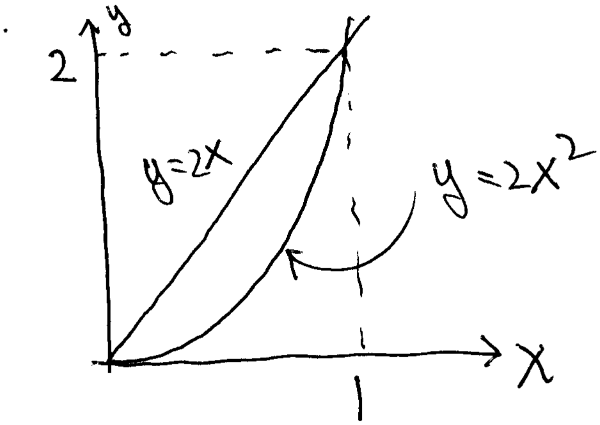
MAT 21D, Temple, Spring 2011

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Let $\mathcal{R}$ denote the region in the (x, y)-plane bounded by the curves $y = 2x^2$ and $y = 2x$.

(a) Sketch the region $\mathcal{R}$.



(b) Set up the iterated integral $\iint_{\mathcal{R}} f(x, y)\, dA$ in the order $dy\, dx$.

$$\int_0^1 \int_{2x^2}^{2x} f(x,y)\,dy\,dx$$

(c) Set up the iterated integral $\iint_{\mathcal{R}} f(x, y)\, dA$ in the order $dx\, dy$.

$$\int_0^2 \int_{y/2}^{\sqrt{y/2}} f(x,y)\,dx\,dy$$

(d) Evaluate the integral $\iint_{\mathcal{R}} xy\, dA$.

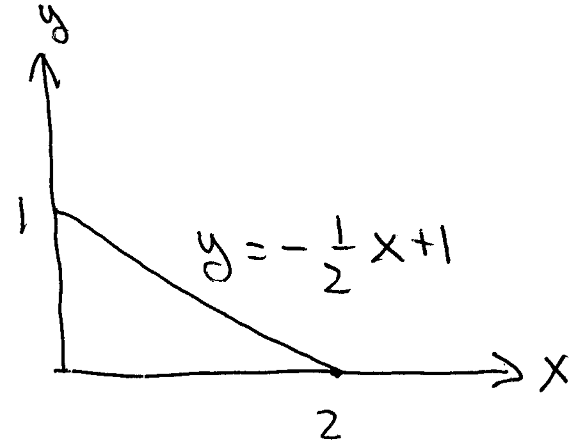
$$\int_0^1 \int_{2x^2}^{2x} xy\,dy\,dx = \frac{1}{2}\int_0^1 xy^2 \Big]_{y=2x^2}^{y=2x} dx = \frac{1}{2}\int_0^1 4x^3 - 4x^5\,dx$$

$$= \cancel{2}\left[\frac{x^4}{\cancel{4}_{2}} - \frac{x^6}{{}_{3}6}\right]_0^1 = \frac{1}{2} - \frac{1}{{}^{2}3} = \frac{3-2}{6} = \frac{1}{6}$$

Problem #2 (20pts): The formula for the kinetic energy stored in a rotating plate $\mathcal{R}$ is $KE = \frac{1}{2} I_L \omega^2$ where $I_L = \int\int_{\mathcal{R}} r^2 \delta(x,y)\,dA$ is the moment of inertia about the axis of rotation L, and ω is the angular velocity in radians per second.

(a) Find the total mass of a metal plate $\mathcal{R}$ of density $\delta(x,y) = x$, bounded by the curve $y = -\frac{1}{2}x + 1$ and the x- and y-axes.

$$\int_0^2 \int_0^{-\frac{1}{2}x+1} x\,dy\,dx = \int_0^2 x y \Big]_{y=0}^{y=-\frac{1}{2}x+1} dx$$



$$= \int_0^2 x\left(-\tfrac{1}{2}x+1\right) dx = \int_0^2 -\frac{x^2}{2} + x\,dx$$

$$= -\frac{x^3}{6} + \frac{x^2}{2}\Big]_0^2 = -\frac{8}{6} + \frac{4}{2} = \frac{4}{6} = \frac{2}{3}$$

(b) Find the moment of inertia about the y-axis of the metal plate in (a).

$$\int_0^2 \int_0^{-\frac{1}{2}x+1} x^3\,dy\,dx = \int_0^2 x^3\left(-\tfrac{1}{2}x+1\right) dx = \int_0^2 -\frac{x^4}{2} + x^3\,dx$$

$$= -\frac{x^5}{10} + \frac{x^4}{4}\Big]_0^2 = -\frac{32}{10} + \frac{16}{4} = -\frac{64}{20} + \frac{80}{20} = \frac{16}{20}$$

$=4/5$

(c) Derive the formula for the radius of gyration R, (the radius at which all the mass should be placed in order to obtain the same kinetic energy), and use it to find the radius of gyration for the axis and plate of part (a).

$$KE = \frac{1}{2} I_y \omega^2 = \frac{1}{2} M R^2 \omega^2$$

$$\frac{1}{\cancel{2}} I_y \cancel{\omega^2} = \frac{1}{\cancel{2}} M R^2 \cancel{\omega^2}$$

$$R = \sqrt{\frac{I_y}{M}}$$

From parts (a), (b) $M = \frac{1}{3}$, $I_y = \frac{3}{20}$ $\Rightarrow$

$$R = \sqrt{\frac{3}{20} \cdot \frac{3}{1}} = \sqrt{\frac{9}{20}}$$

Problem #3 (20pts): Set up the iterated integral in cylindrical coordinates (r, θ, z) for the volume of the right circular cylinder bounded on the sides by the circle

$$(x-1)^2 + y^2 = 1,$$

below by the plane $z = 0$, and above by the plane $z = 3 - x$. Reduce the formula to a single integral in θ alone. (Do not solve!)

$(x-1)^2 + y^2 = 1$ $\qquad x = r\cos\theta$

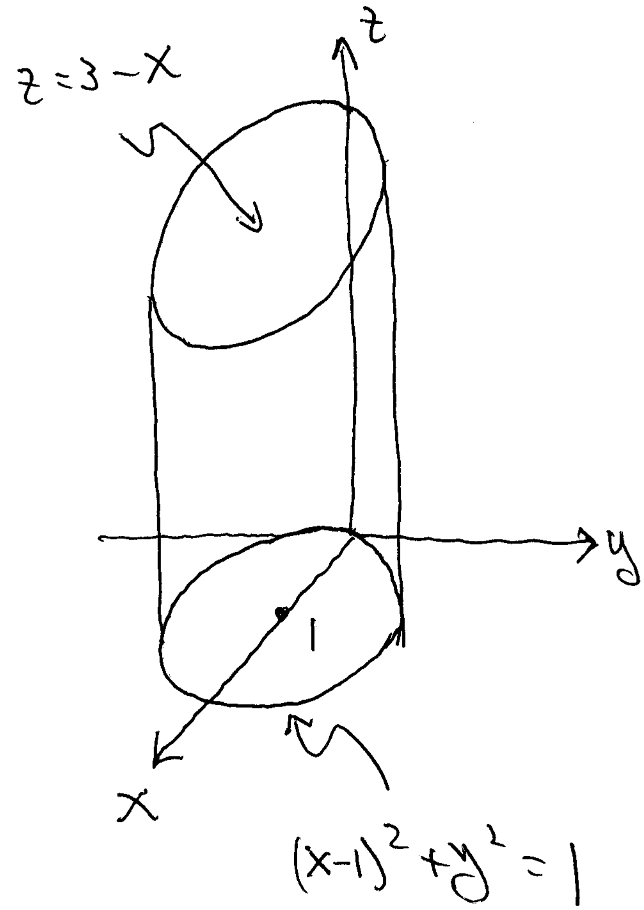
$y = r\sin\theta$

$x^2 - 2x + \cancel{1} + y^2 = \cancel{1}$

$r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta = 0$

$r^{\cancel{2}} = 2\cancel{r}\cos\theta$

$\boxed{r = 2\cos\theta}$

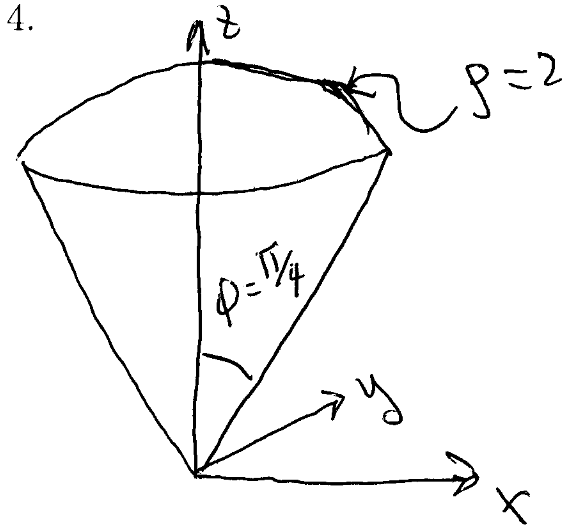


$z = 3 - x = 3 - r\cos\theta$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{3-r\cos\theta} r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r(3 - r\cos\theta)\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left.\frac{3r^2}{2} - \frac{r^3}{3}\cos\theta\right]_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} 6\cos^2\theta - \frac{8}{3}\cos^4\theta\,d\theta \checkmark$$

Problem #4 (20pts): Use spherical coordinates (ρ, ϕ, θ) to find the volume of the (ice cream cone shaped) right circular cone bounded on the sides by $\phi = \pi/4$, and above by the sphere $x^2 + y^2 + z^2 = 4$.



$$\rho = \sqrt{x^2+y^2+z^2} = 2 \Rightarrow$$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \sin\varphi \left.\frac{\rho^3}{3}\right]_0^2 d\varphi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} \left.-\cos\varphi\right]_0^{\pi/4} d\theta = \frac{8}{3}\left(-\frac{\sqrt{2}}{2}+1\right)\cdot 2\pi$$

$$= \frac{16\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

Problem #5 (20pts): Let $\mathcal{R}_{xy}$ denote the area bounded by the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Use the substitution $x = 3u$, $y = 2v$ to evaluate the area of $\mathcal{R}_{xy}$ by integrating over the corresponding region $\mathcal{R}_{uv}$ in uv-coordinates.

$$\iint_{R_{xy}} dA = \iint_{R_{uv}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$$

$$= \iint_{R_{uv}} 6\,du\,dv = \int_0^{2\pi} \int_0^1 6\,r\,dr\,d\theta$$

R_{uv}: $u^2+v^2 \le 1$

$$= \int_0^{2\pi} 6 \left. \frac{r^2}{2} \right]_{r=0}^{r=1} d\theta = 3 \cdot 2\pi$$

$$= \boxed{6\pi}$$