HOMEWORK MATH 22C

(1) A toy rocket takes off at $t = 0$ straight up, $m = 10 \text{kg}$, $F(t) = 40(10 - t)$ is the net force of the rocket (including all other forces except gravity), and $F_g = 9.8$ is the acceleration of gravity, where everything is given in meters, seconds and kg. Write the ODE determined by Newton’s force law $F = ma$, and solve it to find the height of the rocket when the rocket shuts off at $t = 10s$.

(2) Determine the order of the following equations, whether the following equations are linear or nonlinear, and give the initial conditions appropriate for these equations:

(i) $y' + y + 2t = 0$
(ii) $y''' + 2y' - \frac{1}{y} = \sin t$
(iii) $y^{(12)} + y'y = \cos t$
(iv) $y'' = y$

(3) Write the following linear constant coefficient ODE as a first order system of the form $y' = Ay$ where $A$ is a $3 \times 3$ matrix of constants, and $y(t) = (y_1(t), y_2(t), y_3(t))$ is the unknown function $y(t) \in \mathbb{R}^2$:

$$y''' + 2y'' - 4y' + 5y = 0$$

(4) Write the following nonlinear ODE as a first order system:

$$2y''' + t^2y'' + y^2 = 3$$
(5) Consider the nonlinear equation \( y' = a^2 y^2 \):

(i) Find a formula for the solution of the initial value problem \( y' = a^2 y^2 \) with initial condition \( y(t_0) = y_0 \).

(ii) Assuming \( y_0 > 0 \), determine the maximal \( \epsilon \) such that the initial value problem in (i) has a solution for \( t \in (t_0 - \epsilon, t_0 + \epsilon) \).

(iii) Explain why Theorem 1 of Section 3 applies, and why Theorem 2 does not.

(6) Consider the scalar nonlinear ODE
\[ y' = 4(y - 1)(y^2 + 4y + 3). \]
Find the rest points, linearize the equation about the rest points, use the linearized equation to determine their stability, and draw the phase diagram to describe the solutions. Use the phase diagram to justify your conclusion about the stability of the rest points.

(7) Find the rest points, and linearize the equations, find a basis of eigen-solutions of the linearized equations, and determine the stability at each rest point, for the pendulum with friction
\[ \ddot{\theta} + \frac{g}{L} \sin \theta = -k \dot{\theta}. \]

(8) Write the harmonic oscillator
\[ \ddot{\theta} + \frac{g}{L} \sin \theta = 0, \]
as a \( 2 \times 2 \) first order linear system, and show that the real and imaginary parts of the eigensolutions \( \left( \begin{array}{c} 1 \\ -i\alpha \end{array} \right) e^{-i\omega t} \) give the basis of solutions of the harmonic oscillator translated into the notation of the \( 2 \times 2 \) linear system you found.
(9) Find the curves $x(t)$ along which solutions $u(x, t)$ of the linear advection-type equation

$$u_t + xu_x = 0,$$

are constant.

(10) Find formulas for the wave length and frequency of the density wave

$$\rho(x, t) = \rho_0 + \bar{A}\sin b(x - ct),$$

and show that it agrees with an oscillation in the density created at $x = 0$ by a linear oscillator,

$$\rho(t) = \rho_0 + A\sin(at),$$

exactly when $A = -\bar{A}$, $bc = a$.

(Hint: set $x = 0$ and match the solutions. Recall: The wave length is the $x$-length of one period at fixed time, and the frequency at fixed $x$, is $1/\{\text{the time of one period}\}$.)

(11) Find the right-going wave $f(x - ct)$ and the left going wave $f(x + ct)$ created by the solution of the initial value problem

$$u_{tt} - c^2u_{xx} = 0,$$

$$u(x, 0) = \sin(x)$$

$$u_t(x, 0) = 2\sin(x)$$