HOMEWORK MATH 22C

(1) A toy rocket takes off at $t = 0$ straight up, $m = 10\text{kg}$, $F(t) = 40(10 - t)$ is the net force of the rocket (including all other forces except gravity), and $F_g = 9.8$ is the acceleration of gravity, where everything is given in meters, seconds and kg. Write the ODE determined by Newton’s force law $F = ma$, and solve it to find the height of the rocket when the rocket shuts off at $t = 10s$.

(2) Determine the order of the following equations, whether the following equations are linear or nonlinear, and give the initial conditions appropriate for these equations:

(i) $y' + y + 2t = 0$

(ii) $y''' + 2y' - \frac{1}{y} = \sin t$

(iii) $y^{(12)} + y'y = \cos t$

(iv) $y'' = y$

(3) Write the following linear constant coefficient ODE as a first order system of the form $y' = Ay$ where $A$ is a $3 \times 3$ matrix of constants, and $y(t) = (y_1(t), y_2(t), y_3(t))$ is the unknown function $y(t) \in \mathbb{R}^2$:

$$ y''' + 2y'' - 4y' + 5y = 0 $$

(4) Write the following nonlinear ODE as a first order system:

$$ 2y''' + t^2y'' + y^2 = 3 $$
(5) Consider the nonlinear equation \( y' = a^2 y^2 \):

\( \text{(i)} \) Find a formula for the solution of the initial value problem \( y' = a^2 y^2 \) with initial condition \( y(t_0) = y_0 \).

\( \text{(ii)} \) Assuming \( y_0 > 0 \), determine the maximal \( \epsilon \) such that the initial value problem in \( \text{(i)} \) has a solution for \( t \in (t_0 - \epsilon, t_0 + \epsilon) \).

\( \text{(iii)} \) Explain why Theorem 2 of Section 3 applies, and why Theorem 1 does not.

(6) Consider the scalar nonlinear ODE

\[ y' = 4(y - 1)(y^2 + 4y + 3) . \]

Find the rest points, linearize the equation about the rest points, use the linearized equation to determine their stability, and draw the phase diagram to describe the solutions. Use the phase diagram to justify your conclusion about the stability of the rest points.

(7) Consider a 2 \( \times \) 2 constant coefficient linear system of form

\[ \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} . \tag{1} \]

\( \text{(i)} \) Write this as \( \dot{y} = Ay \), and show that if \( (\lambda, R) \) is an eign-pair satisfying \( AR = \lambda R \), then \( y = Re^{\lambda t} \) is a solution. (Keep track of what is a scalar and what is a vector).

(8) Write the pendulum with friction

\[ \ddot{\theta} + \frac{g}{L} \sin \theta = -k \dot{\theta} , \]

as a 2 \( \times \) 2 first order linear system. Linearize the equations at the rest point \( \theta = 0 \). Show that in the case \( k = 0 \), the
eigensolutions are given by \( \left( \frac{1}{-i\alpha} \right) e^{-i\alpha t} \), and the real and imaginary parts of the these complex eigensolution give the basis of solutions of the harmonic oscillator translated into the notation of the 2 \times 2 linear system you found.

(9) Write the characteristic ODE’s, (the equations for the curves along which a solution \( u(x, t) \) is constant) for the equation \( e^{xt} u_t + \sin x^2 tu_x = 0 \). Without solving these equations, use one of the theorems to argue that two solutions cannot intersect in the \( xt \)-plane, hence no shock waves.

(10) Find the curves \( x(t) \) along which solutions \( u(x, t) \) of the linear advection-type equation
\[
u_t + xu_x = 0,
\]
are constant. Are there shock waves?

(11) Consider a solution of the nonlinear Burgers equation
\[
u_t + uu_x = 0
\]
starting from initial data \( u(x, 0) = u_0(x) \) where \( u_0(x) = 1 \) for \( x \leq -1 \) and \( u_0(x) = -1 \) for \( x \geq 1 \). Assume we don’t know what the solution looks like between \(-1 < x < 1\). What is the longest time the solution \( u(x, t) \) could live before a shock wave formed? Explain. For such a solution, describe the solution \( u(x, 1) \). (Hint: trace the important characteristics.)

(12) Find formulas for the wave length and frequency of the density wave
\[
\rho(x, t) = \rho_0 + \bar{A} \sin b(x - ct),
\]
and show that it agrees with an oscillation in the density created at \( x = 0 \) by a linear oscillator,
\[
\rho(t) = \rho_0 + A \sin (at),
\]
exactly when \( A = -\bar{A}, \ bc = a \).
(Hint: set $x = 0$ and match the solutions. Recall: The wave length is the $x$-length of one period at fixed time, and the frequency at fixed $x$, is $1 \div \{\text{the time of one period}\}$.)

**Problem 13**

Find the right-going wave $f(x - ct)$ and the left going wave $f(x + ct)$ created by the solution of the initial value problem

$$u_{tt} - c^2 u_{xx} = 0,$$

$$u(x, 0) = \sin(x)$$

$$u_t(x, 0) = 2\sin(x)$$

********************************************************************************End of Assignment 1**********************************************************************

**Problem 14**

Verify the second half of Theorem 4 of Section 7 for solutions of the nonlinear wave equation: The 2-Riemann invariant $s = u - h(v)$ is constant along 2-characteristic curves $(x(t), t)$ satisfying

$$\dot{x} = \lambda_2 = c.$$

(Hint: Modify the argument given for the 1-characteristics.)

**Problem 15**

Verify the second half of Theorem 6 of Section 8 for solutions of the compressible Euler equations: The 2-Riemann invariant $s = u - \bar{h}(\rho)$ is constant along 2-characteristic curves $(x(t), t)$ satisfying

$$\dot{x} = \lambda_2 = u + c.$$

(Hint: Modify the argument given for the 1-characteristics.)
Recall that for the $p$-system, the function $h(v)$ was defined in Section 7 to satisfy
\[ h'(v) = \sqrt{-p'(v)}. \]
Show that $\bar{h}(\rho) = h(v) = h(1/\rho)$ satisfies the condition
\[ \bar{h}'(\rho) = -\frac{\sqrt{p'(\rho)}}{\rho} \]
assumed in Chapter 8. Conclude that $h$ and $\bar{h}$ are the same functions of the state variables $\rho$ and $u$, and hence the Riemann invariants for the Lagrangian $p$-system are the same functions of $\rho$ and $u$ as the Riemann invariants of the Euler equations.

Show that the eigenfamilies of the $p$-system will be genuinely nonlinear in the sense of Section 8 under the condition that $p''(v) > 0$. 