11–Applications of the Divergence Theorem: The Compressible Euler Equations

MATH 22C

1. Derivation of the Compressible Euler Equations

In this section we use the divergence theorem to derive a physical interpretation of the compressible Euler equations as the continuum version of Newton's laws of motion. Reversing the steps then provides a derivation of the compressible Euler equations from physical principles. The compressible Euler equations are

$$\begin{aligned} \rho_t + Div(\rho\mathbf{u} &= 0 \quad (1)\\ (\rho u^i)_t + Div(\rho u^i \mathbf{u} + p\mathbf{e}^i) &= 0, \quad (2) \end{aligned}$$

where ρ is the density (mass per volume), $\mathbf{u} = (u_i, u_2, u_3)$ is the velocity, and p is the pressure (force per area) in the fluid. For example, and for simplicity, we can assume the pressure is a known function of the density through the equation of state $p = p(\rho)$. A solution of the equations would consist of functions $\rho(\mathbf{x}, t), \mathbf{u}(\mathbf{x}, t)$, $\mathbf{x} = (x, y, z)$ that meets the partial derivative constraints (1) and (2) at every point. The compressible Euler equations describe the motion of a compressible fluid like air, under the assumption that there is no viscosity or dissipation. One can view these as the expressioin of Newton's laws for a continuous media. That is, in words, the first Euler equation (1), (often referred to as the continuity equation), expresses *conservation of mass*, that mass is nowhere created nor destroyed. The second Euler equation, (2), is the continuum version of Newton's force law $\mathbf{F} = m\mathbf{a} = \frac{d(mv)}{dt}$. It expresses that changes of momentum are due solely to gradients in the pressure. It is interesting that when the pressure depends only on the density, $p = p(\rho)$, these two Euler equations stand on their own. But when the pressure depends on the temperature as well, we obtain the full system of compressible Euler equations by including one final equation for the energy, written most simply in the form

$$E_t + Div[(E + p)\mathbf{u}] = 0, \quad (3)$$

where E is the energy per volume. For smooth solutions that are shock free, the energy equation can be replaced by the much simpler entropy equation

$$S_t + Div(S\mathbf{u}) = 0. \quad (4)$$

To make sense of this, we must introduce the second law of thermodynamics and explain the connections between the energy density E, the entropy density S and the pressure p. The explanation is most interesting because it connects up the compressible Euler equations to thermodynamics, and by this we'll see that entropy (a measure of reversibility) can be given a precise definition in the context of a perfect fluid, and we'll see that (4) precisely expresses that entropy does not increase along particle paths when solutions are smooth. As we'll see, this is consistent with the fact that we neglect viscosity and heat conduction in the derivation of (1)-(4). In further developments of the theory of shock waves, (see e.g. [**?**]), it follows that the entropy equation (4) is violated on shock waves, but the condition that entropy increases on shock waves is sufficient to rule out the unphysical rarefaction shocks that satisfy the Rankine-Hugoniot jump conditions.

So in this section we first use the divergence theorem to derive the physical principles expressed by the first two Euler equations (1), (2). When $p = p(\rho)$, this stands on its own. We next derive the continuum version of conservation of energy expressed by the energy equation (3). We then introduce the second law of thermodynamics together with enough fluid dynamics to derive (4) and interpret it as telling us that entropy does not increase when shocks are not present. In the final section we describe the fundamental *polytropic equation of state* that connects the pressure p to the energy E and entropy S for noninteracting gas of molecules. The polytropic equation of state describes a gas of identical molecules each consisting of r atoms.

The compressible Euler equations with polytropic equation of state are a fundamental set of equations. Every term and every constant in the compressible Euler equations with polytropic equation of state is derivable from first principles. Nothing is phenomenological (like a constant whose value is determined by an experiment) or ad hoc (like a term added or a value assigned to make a numerical experiment fit the data). For this reason they are fundamental to Applied Mathematics, Physics and Fluid Mechanics, and they provide the main physical setting for the Mathematical Theory of Shock Waves. A student who learns this has the opportunity to connect up thermodynamics, fluid mechanics, physics, and PDE's in a unified, self-contained, fundamental theory. The gain is well worth the effort!

2. The Mass and Momentum Equations

To start, restrict attention first to the first two Euler equations equations (1) and (2). To derive the physical principles that underly the equations, choose any fixed volume $\mathcal{V}$, integrate the equations (1), (2) over $\mathcal{V}$, and apply the divergence theorem. Start with (1), the so called *continuity equation*:

$$0 = \int\int\int_{\mathcal{V}} \rho_t + Div(\rho\mathbf{u})\, dV = \int\int\int_{\mathcal{V}} \rho_t\, dV + \int\int\int_{\mathcal{V}} Div(\rho\mathbf{u})\, dV. \tag{5}$$

Now in the first integral, since the integration is over $\mathbf{x}$ not t, we claim we can pass the partial derivative with respect to t out through the integral sign to get a regular derivative on the outside,

$$0 = \int\int\int_{\mathcal{V}} \rho_t\, dV = \frac{d}{dt}\int\int\int_{\mathcal{V}} \rho\, dV. \tag{6}$$

To verify this, use the definition of derivative directly: the definition of

$$\frac{d}{dt}\int\int\int_{\mathcal{V}} \rho\, dV$$

leads directly to

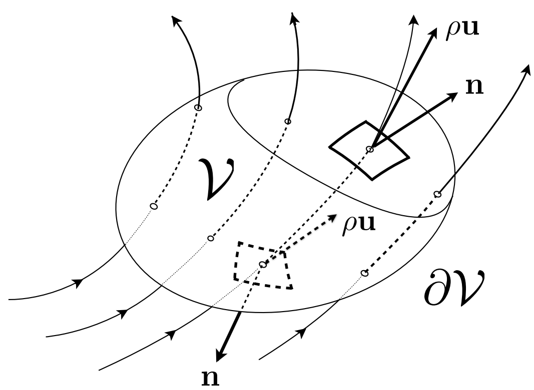
$$\begin{aligned}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int\int\int_{\mathcal{V}} \rho(\mathbf{x}, t+\Delta t)\, dV_{\mathbf{x}} - \int\int\int_{\mathcal{V}} \rho(\mathbf{x}, t+\Delta t)\, dV_{\mathbf{x}}\right\}& \\ = \lim_{\Delta t\to 0}\int\int\int_{\mathcal{V}} \frac{\rho(\mathbf{x}, t+\Delta t) - \rho(\mathbf{x}, t)}{\Delta t}\, dV_{\mathbf{x}}& \\ = \int\int\int_{\mathcal{V}} \rho_t(\mathbf{x}, t)\, dV_{\mathbf{x}}.&\end{aligned}$$

To the second integral in (7) we apply the Divergence theorem to obtain:

$$\int\int\int_{\mathcal{V}} Div(\rho\mathbf{u})\, dV = \int\int_{\partial\mathcal{V}} \rho\mathbf{u}\cdot\mathbf{n}\, dA. \tag{7}$$

Putting these together we obtain:

$$\underbrace{\frac{d}{dt}\int\int\int_V \rho\, dv}_{\substack{\text{Total Mass}\\ \text{Inside } \mathcal{V}}} = \underbrace{-\int\int_{\partial V} \rho\mathbf{u}\cdot\mathbf{n}\, dA}_{\substack{\text{Mass Flux}\\ \text{Out Through } \partial\mathcal{V}}}$$



Conservation of Mass: The time rate of change of the mass in volume $\mathcal{V}$ equals minus the mass flux through the boundary $\partial\mathcal{V}$

Conclude: The continuity equation implies that *the total rate of change of mass inside any volume is equal to minus the flux of mass out through the boundary*, a precise expression of conservation of mass for that volume. Since the volume $\mathcal{V}$ is arbitrary, it follows that mass is conserved in every volume, hence mass is conserved. But *reversing the steps*, we have a derivation the continuity equation as an expression of *conservation of mass*. That is, we can *start* by *defining* the physical principle of conservation of momentum as meaning precisely that “the time rate of change of the total mass in every volume is equal to minus the flux of mass through the boundary" in the sense that (7) holds for *every* volume V, and from this, using the divergence theorem in reverse, we can conclude that (7) holds in every volume, and from this (using that the integral of a continuous function is zero in every volume iff the function is identically zero) we can conclude that conservation of mass in this sense implies the continuity equation (1). We can now take this reversed argument as a physical derivation of (1) from a precise physical expression of the principle of conservation of mass in a continuous media. (Of course, the derivation technically is only valid for continuous functions, so shock waves are another matter!)

Consider next the second Euler equation (2),

$$(\rho u^i)_t + Div(\rho u^i \mathbf{u} + p\mathbf{e}^i) = 0,$$

which is really three equations, one for each $i = 1, 2, 3$. We now use the divergence theorem to show that it implies conservation of momentum in every volume. That is, we show that the time rate of change of momentum in each volume is minus the flux through the boundary minus the work done on the boundary by the pressure forces. This is the physical expression of Newton's force law for a continuous medium. For this, integrate (2) over a fixed volume $\mathcal{V}$ to obtain

$$0 = \int\int\int_{\mathcal{V}} (\rho u^i)_t \, dV + \int\int\int_{\mathcal{V}} Div(\rho u^i \mathbf{u}) \, dV + \int\int\int_{\mathcal{V}} Div(p\,\mathbf{e}^i) \, dV. \tag{8}$$

Taking the time derivative out through the first integral and applying the divergence theorem to the second two integrals (as we did for the continuity equation) we obtain

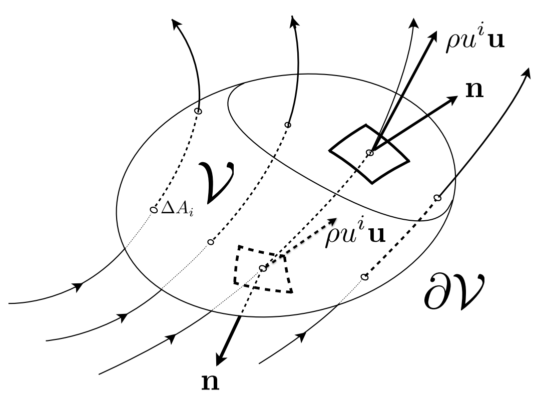
$$\frac{d}{dt}\int\int\int_{\mathcal{V}} (\rho u^i) = -\int\int_{\partial V} (\rho u^i)\mathbf{u}\cdot\mathbf{n}\, dA - \int\int_{\partial V} p\,\mathbf{e}^i\cdot\mathbf{n}\, dA. \tag{9}$$

Here $\rho u^i \mathbf{u}$ is the i-momentum flux vector, with dimensions

$$(\rho u^i)\cdot\mathbf{u} = \frac{mass \times velocity}{volume}\cdot\frac{distance}{time} = \frac{i - momentum}{area\ time},$$

so minus the first integral on the right hand side of (9) integrates over the area to give minus the rate at which i-momentum is passing out through the boundary $\partial\mathcal{V}$ of $\mathcal{V}$. Similarly, for the second integral on the right hand side of (9), the dimensions of p is force per area, $\mathbf{e}^i \cdot \mathbf{n} = n^i$ equals the i'th component of the normal, so minus the second integral on the right hand side of (9) integrates over the area to give minus the total i-component of the pressure force on the boundary $\partial\mathcal{V}$. Putting these together we obtain:

$$\underbrace{\frac{d}{dt}\int\int\int_V \rho u^i\, dv}_{\substack{\text{Total } i\text{-Momentum}\\ \text{Inside } \mathcal{V}}} = \underbrace{-\int\int_{\partial V} (\rho u^i)\mathbf{u}\cdot\mathbf{n}\, dA}_{\substack{i\text{-Momentum Flux}\\ \text{Out Through } \partial\mathcal{V}}} \underbrace{-\int\int_{\partial V} p\,\mathbf{e^i}\cdot\mathbf{n}\, dA}_{\substack{\text{Total } i\text{-Component}\\ \text{of the Pressure Force}\\ \text{Exerted on } \partial\mathcal{V}}}$$



Conservation of Momentum: The time rate of change of Momentum in a given volume $\mathcal{V}$ is minus the i-momentum flux out through the boundary $\partial\mathcal{V}$ minus the total force of the pressure exerted on $\partial\mathcal{V}$.

Conclude: The i-momentum equation implies that *the total rate of change of i-momentum inside any volume is equal to minus the flux of i-momentum out through the boundary minus the net force due to the pressure exerted on the boundary*, a precise expression of conservation of i-momentum for that volume. Note that the continuum version of Newton's force law $\mathbf{F} = m\mathbf{a}$ appears as an integrated form of $\mathbf{F} = \frac{d(m\mathbf{v})}{dt}$, that is, on the LHS $\mathbf{F}$ is replaced by the net force of the pressure on the boundary $\partial\mathcal{V}$, and the right hand side is replaced but the total time rate of change of i-momentum in $\mathcal{V}$. Since the volume $\mathcal{V}$ is arbitrary, it follows that the balance of i-momentum holds in every volume, $i = 1, 2, 3$, and we say that *momentum is conserved*. Again, we can *reverse the steps*, and *start* by defining the physical principle of conservation of momentum as meaning precisely that "the time rate of change of momentum in every volume is equal to minus the flux of momentum through the boundary minus the force due to the pressure on the boundary" in the sense that (9) holds for *every* volume V, and from this, using the divergence theorem in reverse, we can conclude that (8) holds in every volume, and from this (using again that the integral of a continuous function is zero in every volume iff the function is identically zero) we can conclude that the momentum equations (2) for each $i = 1, 2, 3$, follows from the principle of conservation of momentum as expressed in (9). We can now take this reversed argument as a physical derivation of (2) from a precise physical expression for the principle of conservation of momentum in a continuous media. (Again, the derivation technically is only valid for continuous functions, so shock waves are another matter!)

3. The energy equation

To complete the theory of the compressible Euler equations to the case when p is not a function of the density alone, we consider finally the energy equation, the fifth and last equation in the Euler system. The equation couples to (1), (2) through the pressure when the pressure depends on the density as well as the temperature, specific energy e or specific entropy s, say $p = p(\rho, s)$. The energy equation is

$$E_t + Div((E + p)\mathbf{u}) = 0, \tag{10}$$

where

$$E = \rho e + \frac{1}{2}\rho|\mathbf{u}|^2 \tag{11}$$

is the total energy per volume, the sum of the internal energy per volume ρe (e=$e(\rho, s)$=specific internal energy=the energy per mass stored in the vibrations of the molecules so that ρe is internal energy per volume), and the kinetic energy $\frac{1}{2}\rho|\mathbf{u}|^2$, the kinetic energy due to the motion of the fluid particles. To finish we use the divergence theorem to show that (10) implies conservation of energy in every volume. That is, we show that the time rate of change of energy in each volume is minus the flux of energy out through the boundary minus the net rate at which work is done by the pressure force acting on the boundary. This is the continuum version of Newton's principle that the time rate of change of work should be equal to the energy change per time. That is, if $\mathbf{F}$ is a classical force, then

$$Work = Force \times Displacement$$

gives

$$\frac{d}{dt}Work = \frac{d}{dt}\mathbf{F}\cdot d\mathbf{s} = \frac{d}{dt}m\frac{d\mathbf{v}}{dt}d\mathbf{s} = \left(m\frac{d\mathbf{v}}{dt}\frac{d\mathbf{s}}{dt}\right) = \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}^2\right) = \frac{d}{dt}Energy$$

but in the continuum case, the work per time done by the pressure force can be stored in the internal energy of vibration as well as in the kinetic energy of motion. To see this, integrate (10) over a fixed volume $\mathcal{V}$ to obtain

$$0 = \int\int\int_{\mathcal{V}} E_t\, dV + \int\int\int_{\mathcal{V}} Div(E\mathbf{u})\, dV + \int\int\int_{\mathcal{V}} Div(p\,\mathbf{u})\, dV. \tag{12}$$

Taking the time derivative out through the first integral and applying the divergence theorem to the second two integrals (as we did for the momentum equation) we obtain

$$\frac{d}{dt}\int\int\int_{\mathcal{V}} E = -\int\int_{\partial V} E\mathbf{u}\cdot\mathbf{n}\, dA - \int\int_{\partial V} p\mathbf{u}\cdot\mathbf{n}\, dA. \tag{13}$$

Now $E\mathbf{u}$ is the Energy flux vector, with dimensions

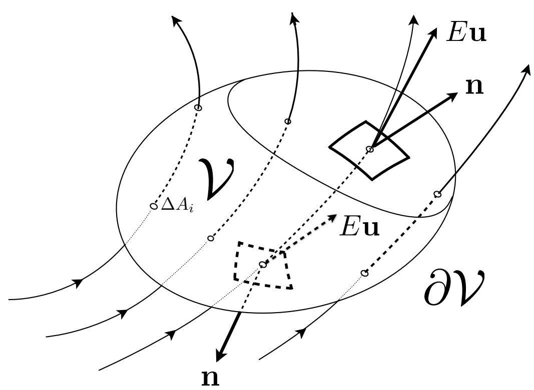
$$E\mathbf{u} = \frac{energy}{volume}\cdot\frac{distance}{time} = \frac{energy}{area\ time},$$

so minus the first integral on the right hand side of (9) integrates over the area to give minus the rate at which Energy is passing out through the boundary $\partial\mathcal{V}$ of $\mathcal{V}$. Consider then the second integral on the right hand side of (13). The dimensions of p are force per area (as always), so $p\mathbf{u}$ has dimensions

$$p\mathbf{u} = \frac{force}{area}\cdot\frac{distance}{time} = \frac{force \times distance}{area \times time} = \frac{work}{area\ time},$$

and so minus the second integral on the right hand side of (9) integrates over the area to give the total work per time done by the pressure force acting on the boundary $\partial\mathcal{V}$ of $\mathcal{V}$. Putting these together we obtain:

$$\frac{d}{dt}\underbrace{\int\int\int_V E\,dv}_{\substack{\text{Total Energy}\\ \text{Inside }\mathcal{V}}} = -\underbrace{\int\int_{\partial V} E\mathbf{u}\cdot\mathbf{n}\,dA}_{\substack{\text{Energy Flux}\\ \text{Out Through }\partial\mathcal{V}}} - \underbrace{\int\int_{\partial V} p\mathbf{u}\cdot\mathbf{n}\,dA}_{\substack{\text{Rate of Work Done}\\ \text{by Pressure Force}\\ \text{On }\partial\mathcal{V}}}$$



Conservation of Energy: The time rate of change of energy in each volume is minus the flux of energy out through the boundary minus the net rate at which work is done by the pressure force acting on the boundary.

Conclude: The energy equation implies that *the total rate of change of Energy inside any volume is equal to minus the flux of Energy out through the boundary minus the net rate at which work is done by the pressure force on the boundary*, a precise expression of conservation of energy for that volume. Note that the continuum version of Newton's power law $\mathbf{F}\cdot\mathbf{u} = \frac{dW}{dt}$ appears with $\mathbf{F}\cdot\mathbf{u}$ replaced by $p\mathbf{n}\cdot\mathbf{u}$ where $p\mathbf{n}$ is the force per area, implying $p\mathbf{n}\cdot\mathbf{u}$ is the work per area time, so that integrating out the area gives the work per time. Since work is energy, work per time appropriately contributes to balance the time rate of change of the energy expressed in the other two terms in (13). Thus the second term on the RHS of (13) represents the pressure forces doing work on the fluid, and the balance of energy law (13) tells us that the last Euler equation (10) expresses that the pressure is the *only* force doing work in the fluid, thereby completing the continuum version of Newton's law of energy conservation. Since the volume $\mathcal{V}$ is arbitrary, it follows that the balance of Energy holds in every volume as a consequence of (10), and we say that *energy is conserved*. Again, we can *reverse the steps*, and *start* by defining the physical principle of conservation of energy as meaning precisely that "the time rate of change of energy in every fixed volume is equal to minus the flux of energy out through the boundary minus the rate at which the pressure force does work on the boundary" in the sense that (13) holds for *every* volume V, and from this, using the divergence theorem in reverse, we can conclude that (10) holds in every volume. From this (using again that the integral of a continuous function is zero in every volume iff the function is identically zero) we can conclude that the energy equation (2) for each $i = 1, 2, 3$, follows from the principle of conservation of energy as expressed in (13). We can now take this reversed argument as a physical derivation of (10) from a precise physical expression for the principle of conservation of energy for a continuous media. (Again, the derivation technically is only valid for continuous functions, so shock waves are another matter!)

Finally, it is interesting that although the derivation of the equations is extremely interesting for understanding how Newton's laws correctly extend to a continuous medium, the derivation has helped us almost *not at all* in understanding the mathematical theory of the evolution of the compressible Euler equations that express these laws.

4. The material derivative and the entropy equation

Entropy enters the theory of the compressible Euler equations through the *Second Law of Thermodynamics*. In this section we derive the entropy equation (4) using the collection of results we already have, together with the Second Law of Thermodynamics. Entropy plays a fundamental role in the theory of shock waves because shock waves introduce loss of information and increase of entropy, and entropy considerations are required to pick out the physical vs the unphysical shock waves. The specific entropy s, the *entropy per mass* of the fluid, is the fifth thermodynamic variable that can be taken in place of any one of ρ, p, e and T. The starting assumption of thermodynamics is that all of these five state variables can be written as a function of any two of them. For example, the pressure is often taken to be a function of the specific volume v and entropy s, $p = p(v, s)$, $v = 1/\rho$.

The entropy as a state variable enters through the Second Law of Thermodynamics, which for our purposes states simply that

$$de = Tds - pdv \tag{14}$$

is an exact differential, or, solving for ds, that

$$ds = \frac{de}{T} + \frac{pdv}{T} \tag{15}$$

is an exact differential. This (15) is exact means no more or less than that the right hand side comes from the gradient of a function, which means there is a function $s(e, v)$ such that

$$\frac{\partial s}{\partial e} = 1/T, \quad \frac{\partial s}{\partial v} = p/T. \tag{16}$$

That is, recall from vector calculus that the line integral of a *differential* like $\frac{de}{T} + \frac{pdv}{T}$ is independent of path if and only if the differential is exact, and $\frac{de}{T} + \frac{pdv}{T}$ is exact just means there exists a function $s(e, v)$ such that (16) holds. Now a class in thermodynamics would show by Carnot cycles that if $\frac{de}{T} + \frac{pdv}{T}$ were not exact, then line integrals around closed curves would not all be zero, and by this one could construct a Carnot cycle around which energy would be created from nothing. Thus, that $\frac{de}{T} + \frac{pdv}{T}$ is exact is equivalent to saying there are no perpetual motion machines. But for us, we can start by taking the second law as simply saying that $ds = \frac{de}{T} + \frac{pdv}{T}$ is an exact differential, or equivalently, solving for ds. To derive (4) from the second law, we first introduce the material derivative.

Definition 1. *Let $\mathbf{u}(\mathbf{x}, t)$ be a given velocity field, and let $f(\mathbf{x}, t)$ be any other function of $\mathbf{x} = (x^1, x^2, x^3)$ and t. Then the material derivative of f associated with velocity $\mathbf{u}$ is defined to be*

$$\frac{Df}{Dt} = f_t(\mathbf{x}, t) + \nabla_{\mathbf{x}} f(\mathbf{x}, t). \tag{17}$$

The material derivative represents the derivative of f *along the particle path*. That is, if the curve $\mathbf{x}(t)$ is a particle path in the sense that it solves the ODE

$$\dot{\mathbf{x}} = u(\mathbf{x}(t), t),$$

then

$$\frac{d}{dt} f(\mathbf{x}(t), t) = \nabla_{\mathbf{x}} f \cdot \dot{\mathbf{x}} + f_t = \frac{Df}{Dt}.$$

The main identity used to connect up the material derivative with the compressible Euler equations is the following (easily verified)

$$Div(f\mathbf{u}) = fDiv(\mathbf{u}) + \nabla f \cdot \mathbf{u}, \tag{18}$$

which holds for any vector field $\mathbf{u}$. Using these we have the following useful theorem:

Theorem 2. *Assume $\rho = \rho(\mathbf{x}, t)$ and $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ solve the the continuity equation (1), and let $f = f(\mathbf{x}, t)$ be any smooth function. Then*

$$(\rho f)_t + Div(\rho f\mathbf{u}) = \rho\frac{Df}{Dt}. \tag{19}$$

Proof: Using the product rule for partial derivatives on the first term and (18) on the second term on left of (19) gives

$$(\rho f)_t + Div(\rho f\mathbf{u}) = [\rho_t + Div(\rho\mathbf{u})]f + \rho\,(f_t + \nabla f \cdot \mathbf{u}) = \rho\frac{Df}{Dt}, \tag{20}$$

as claimed, where the first term vanishes by the continuity equation. $\Box$

Theorem 3. *For smooth solutions, the continuity equation (1) is equivalent to either of the following two equations:*

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -Div(\mathbf{u}), \tag{21}$$

or

$$\frac{1}{v}\frac{Dv}{Dt} = Div(\mathbf{u}). \tag{22}$$

and the momentum equation (2) is equivalent to

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p. \tag{23}$$

Note that (22) gives meaning to the Divergence of a vector field: Namely, the divergence of a vector field gives the rate at which volumes change per volume along the flow of the vector field. Equation (24) gives a continuum version of Newton's force law: Namely, "*mass times acceleration along particle paths is equal to the gradient of the pressure*", telling us that only the gradient of the pressure contributes to changes in the momentum.

Proof: For (21) use (19) to write

$$0 = \rho_t + Div(\rho\mathbf{u}) = \rho_t + \nabla\rho \cdot \mathbf{u} + \rho Div(\mathbf{u}) = \frac{D\rho}{Dt} + \rho Div(\mathbf{u})$$

which gives (21). To obtain (22) from this write

$$\frac{Dv}{Dt} = \frac{D\frac{1}{\rho}}{Dt} = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}Div(\mathbf{u}) = vDiv(\mathbf{u}).$$

This is a complete proof. □

Theorem 4. *Assuming the continuity equation (1), the momentum equation (2) is equivalent to*

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p. \tag{24}$$

Proof: Write (2) in the vector form

$$(\rho\mathbf{u})_t + Div(\rho\mathbf{u}\mathbf{u}^T + pI) = 0,$$

where I is the 3×3 identity matrix and treating $\mathbf{u}$ as a column vector and its transpose $\mathbf{u}^T$ as a row vector, $\mathbf{u}\mathbf{u}^T$ is the 3×3 rank-1 matrix with row column entries $(u^i u^j)$. Then distributing the Div gives

$$0 = (\rho\mathbf{u})_t + Div(\rho\mathbf{u}\mathbf{u}^T) + Div(pI) = \rho\frac{D\mathbf{u}}{Dt} + \nabla p,$$

where we have applied (19) and rewritten $DivpI = \nabla p$. This confirms (24). □

Theorem 5. *Assuming the continuity equation (1) together with the momentum equation (2), the energy equation (3) is equivalent to the equation*

$$\frac{De}{Dt} = -p\frac{Dv}{Dt} = 0. \tag{25}$$

Proof: Now by (11) and (18),

$$E_t + Div(E\mathbf{u}) = \rho\frac{DE}{Dt} = \rho\frac{De}{Dt} + \rho\mathbf{u}\frac{D\mathbf{u}}{Dt}, \tag{26}$$

and

$$Div(p\mathbf{u}) = \nabla p\cdot\mathbf{u} + pDiv(\mathbf{u}).$$

Putting these together and using (24) to cancel the $\nabla p\cdot\mathbf{u}$ term gives

$$\rho\frac{De}{Dt} = -pDiv(\mathbf{u}) = -p\frac{1}{v}\frac{Dv}{Dt},$$

which readily gives (26). □

Theorem 6. *Assuming the continuity equation (1) together with the momentum equation (2), the energy equation (3) is equivalent to the entropy equation (4), namely,*

$$S_t + Div(S\mathbf{u}) = 0, \tag{27}$$

which is equivalent to

$$\frac{Ds}{Dt} = 0 \tag{28}$$

Proof: Starting with the Second Law in the form (15),

$$Tds = de + pdv,$$

it is not difficult to show that

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + p\frac{Dv}{Dt} = 0,$$

where we have applied the energy equation in the form (26). This establishes (28). But since $S = \rho s$ is the entropy density, we can write

$$S_t + Div(S\mathbf{u}) = (\rho s)_t + Div(\rho s\mathbf{u}) = \rho\frac{Ds}{Dt} = 0,$$

as claimed in (28). □

5. The compressible Euler Equations with Polytropic Equation of State

We end this section by summarizing the full compressible Euler equations for a polytropic equation of state. The complete system of compressible Euler equations takes the following conservation form:

$$\begin{aligned} \rho_t + Div(\rho\mathbf{u} &= 0 && (29)\\ (\rho u^i)_t + Div(\rho u^i\mathbf{u} + p\mathbf{e}^i) &= 0, && (30)\\ E_t + Div[(E+p)\mathbf{u}] &= 0, && (31) \end{aligned}$$

with

$$E = \rho e + \frac{1}{2}\mathbf{u}^2, \tag{32}$$

$$\mathbf{u}^2 \equiv |\mathbf{u}|^2 = (u^1)^2 + (u^2)^2 + (u^3)^2.$$

The system (29)-(30) is a system of five equations in the six unknown functions of $(\mathbf{x}, t)$

$$u^1, u^2, u^3, \rho, p, e,$$

consisting of the three components of the velocity vector $\mathbf{u} = (u^1, u^2, u^3)$ (we use superscript indices to be consistent with Einstein summation convention whereby vector components are alway up, c.f. [**?**]), and the

the remaining three variables, the density ρ, pressure p and specific internal energy e are the so called thermodynamic variables. The other two important thermodynamic variables in shock wave theory are the temperature T and the specific entropy s (we'll discuss the entropy in the next section). A principle of thermodynamics is that all of the five thermodynamic variables can be expressed as a function of any two of them. Since by any choice of the two independent thermodynamic variables, there remain six unknowns and five equations, an equation of state which gives the pressure in terms of two other thermodynamic variables, must be given to reduce the number of unknowns by one and thereby close the equations.

The most fundamental equation of state is the so called *polytropic* equation of state given by

$$p = p(\rho, e) = (\gamma - 1)\rho e, \tag{33}$$

where γ is the so called *adiabatic constant* of the gas. Writing $p = p(\rho, e)$ in (29)-(31) closes the compressible Euler equations into a system of five equations in the five unknowns

$$(u^1, u^2, u^3, \rho, e).$$

To write the system in the form of a system of conservation laws

$$U_t + f(U)_x = 0,$$

define the conserved quantities

$$U = (\rho, G, E),$$

using

$$G = (G_1, G_2, G_3) = (\rho u^1, \rho u^2, \rho u^3) = \rho\mathbf{u},$$

and find expressions for (p, e) in terms of U. For example, by (32),

$$e = \frac{E}{\rho} - \frac{1}{2}|\mathbf{u}|^2 = \frac{U_5}{U_1} - \frac{1}{2}\left|\frac{G}{U_1}\right|^2,$$

and so

$$p = (\gamma - 1)\rho e = (\gamma - 1)U_1\left(\frac{U_5}{U_1} - \frac{1}{2}\left|\frac{G}{U_1}\right|^2\right).$$

A final word on the adiabatic gas constant. The polytropic equation of state describes a gas of identical molecules each consisting of r atoms. In this case, assuming the ideal gas law

$$pv = RT, \tag{34}$$

with $v = 1/\rho$ the specific volume and R the universal gas constant, together with the assumption that the internal energy e distributes equally among all the vibrational degrees of freedom, leads logically to a dervivation of (33) together with the relations, c.f. [**?**].

$$e = \frac{R}{\gamma - 1} T, \tag{35}$$

$$\gamma = 1 + \frac{2}{3r}. \tag{36}$$

In particular, for a polytropic gas (33), equation (35) tells us that the internal energy is proportional to the temperature, with proportionality constant $C_V = R/(\gamma - 1)$ called the *specific heat at constant volume*, (the heat required to raise a unit mass one degree); and (36) gives the adiabatic gas constant as a function of the number of atoms r in the gas molecules. Taking the limits $r = 1$ and $r \to \infty$ gives the bounds

$$1 < \gamma \leq \frac{5}{3},$$

the value $\gamma = 5/3$ applying to a mono-tonic gas, and $\gamma \to 1$ in the limit of very heavy molecules. In particular, air is mostly Nitrogen N_2, giving a value of gamma equal to

$$\gamma = 4/3.$$

Conclude: Every term and constant in the compressible Euler equations with polytropic equation of state is derivable from first principles. Nothing is phenomenological (like a constant whose value is determined by an experiment) or ad hoc (like a term added or a value assigned to make a numerical experiment fit the data). For this reason the compressible Euler equations with polytropic equation of state are a fundamental set of equations for Applied Mathematics and Physics, they anchor the subject of PDE's by marking the starting point for Fluid Mechanics, and as such they provide the main physical setting for the Mathematical Theory of Shock Waves.

6. Entropy of a Polytropic gas

We now show how to integrate the second law to find a formula for s as a function of v and T, and thereby show how entropy enters the formulas for a polytropic gas. By this we can find s as a function of any other two thermodynamical variables among ρ, p, e, T.

Theorem 7. *Assume (33)-(36) for a polytropic gas, and assume (14) is an exact differential, so (16) holds. Then*

$$s = c_v \ln v^{\gamma-1} T. \tag{37}$$

Proof: To integrate (14), (meaning to find a function $e(s, v)$ that meets conditions (16)), we introduce a clever change of variables. For this, define what has come to be known as the *free energy*.

$$\psi = e - sT.$$

For us the free energy is important simply because we can solve the second law when expressed in terms of ψ. For this purpose, take differentials on both sides to obtain

$$d\psi = de - sdT - Tds.$$

Using the second law $Tds = de + pdv$, continue

$$d\psi = de - sdT - Tds = de - sdT - de - pdv = -sdT - pdv.$$

or

$$d\psi = -sdT - pdv. \tag{38}$$

We now solve for the value of ψ that makes (38) exact, namely, we find ψ such that

$$\frac{\partial}{\partial T}\psi(T, v) = -s, \quad \text{and} \quad \frac{\partial}{\partial v}\psi(T, v) = -p. \tag{39}$$

But using the idea gas law (34)

$$p(T, v) = \frac{RT}{v},$$

the second equation in (39) becomes

$$\frac{\partial}{\partial v}\psi(T, v) = -\frac{RT}{v}.$$

Remarkably, holding we can anti-differentiate this with respect to v holding T fixed to obtain

$$\psi(T, v) = -RT \ln v + g(T),$$

where $g(T)$ is the arbitrary function of integration. To determine $g(T)$, use the first equation in (39) to write

$$s = -\frac{\partial}{\partial T}\psi(T, v) = -\left(R \ln v + g'(T)\right). \tag{40}$$

But by the definition of ψ we also have

$$c_v T = e = \psi(T, v) + sT = -RT \ln v + g(T) + RT \ln v - Tg'(T), \tag{41}$$

the latter two terms coming from

$$sT = -T\frac{\partial \psi}{\partial T}.$$

Canceling the first and third term on the RHS of (41) gives

$$c_V T = g(T) - Tg'(T),$$

which upon differentiating both sides with respect to T gives

$$c_v = -Tg''(T).$$

By this we obtain a formula for $g''(T)$, namely

$$g''(T) = -\frac{c_v}{T},$$

which upon integrating once gives

$$g'(T) = -c_v \ln T + const.$$

Using this in (40), and setting const. equal to zero, (only changes in entropy are measurable anyway), we obtain

$$s = c_v \ln{(v^{\gamma-1}T)},$$

as claimed. $\Box$

Finally, solving $s = c_v \ln{(v^{\gamma-1}T)}$ for T gives

$$T = v^{1-\gamma} \exp{(\frac{s}{c_v})};$$

and using this in $e = c_v T$ yields the formula

$$e = c_v \frac{1}{v^{\gamma-1}} \exp{(\frac{s}{c_v})} \equiv e(s, v).$$

This together with the second law gives the equation of state of a polytropic gas in terms of (s, v):

$$p = -\frac{\partial e}{\partial v}(s, v) = c_v(\gamma - 1)\frac{1}{v^\gamma} \exp{(\frac{s}{c_v})} \equiv p(s, v).$$

This is the form of the equation of state of a polytropic gas often quoted in the literature. In particular, replacing $v = 1/\rho$ gives primarily because the speed of sound c is given by the formula

$$p(\rho, s) = c_v(\gamma - 1)\rho^\gamma \exp{(\frac{s}{c_v})},$$

and it turns out the correct generalization of the speed of sound σ when p depends on s as well as ρ is

$$\sigma = \sqrt{\frac{\partial p}{\partial \rho}(\rho, s)}.$$

We could obtain this by linearizing the equations just like we did for the barotropic equation of state $p = p(\rho)$ before.

The following important theorem gives the equation for the entropy:

Theorem 8. *Assume the compressible Euler equations with polytropic equation of state. Then the entropy satifies the additional conservation law*

$$S_t + Div(S\mathbf{u}) = 0,$$

where

$$S = \rho s$$

is the entropy per volume, or entropy density. (Note that multiplication by ρ = mass/vol converts s=specific entropy=entropy/mass to S=entropy/vol.)

Proof: Write

$$\begin{aligned}(\rho s)_t + Div(\rho s\mathbf{u}) &= \rho_t s + \rho s_t + sDiv(\rho\mathbf{u}) + \rho\nabla s \cdot \mathbf{u} \\ &= s(\rho_t + div(\rho\mathbf{u}) + \rho\end{aligned}$$

Conclude: The entropy is the state variable that is defined by assuming the second law $de = Tds - pdv$ is an exact differential.

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7. Derivation of the Rankine-Hugoniot Jump Conditions

In this section we apply the Divergence Theorem to derive the Rankine-Hugoniot (RH) jump conditions, which we already used to derive the shock curves of the p-system. So consider a system of conservation laws

$$U_t + f(U)_x = 0, \tag{42}$$

where U and $f(U)$ are vectors in $\mathcal{R}^n$,

$$U = (U_1, ..., U_n),$$

$$f(U) = (f_1(U), ..., f_n(U)).$$

The RH-jump conditions apply to solutions $U(x,t)$ that are smooth on either side a curve $(x(t),t)$ in the (x,t)-plane, but suffers a jump discontinuity across the curve. So assume the shock curve $(x(t),t)$ has finite speed $s = \dot{x}(t)$, and assume that $U(x,t) = U_L(x,t)$ to the left of this curve $x < x(t)$, and $U(x,t) = U_R(x,t)$ to the right of this curve $x > x(t)$, where U_L and U_R are smooth functions that solve the conservation law (42). Now U cannot satisfy (42) on the shock curve because it suffers a jump discontinuity there, so the derivatives are not defined. Even so, the RH condition says that the jump in a solution across a shock wave must be related to the speed of the shock by the condition

$$s[U] = [f(U)]. \tag{43}$$

Here, $[\cdot]$ around a quantity denotes the jump in that quantity from left to right across the shock wave, so $[U] = U_R - U_L$, $[f(U)] = f(U_R) - f(U_L)$, etc. Recall that for the p-system, $U = (v, u)$, and for fixed state U_L and constant s, the jump condition was satisfied by two curves $\mathcal{S}_1(U_L)$ and $\mathcal{S}_2(U_L)$ that had C^2 contact with the rarefaction curves $\mathcal{R}_1^+(U_L)$ and $\mathcal{R}_2^+(U_L)$ at U_L. We then defined the wave curves $W_i(U_L) = \mathcal{S}_1^-(U_L) \cup \mathcal{S}_1^-(U_L)$, $i = 1, 1$, consisting the the states U_R that could be connected to U_L by a shock wave of speed $s = s(U_L, U_R)$ determined by the analysis. So the question remains: where did the Rankine-Hugoniot jump conditions come from?

We now derive the RH conditions starting with the notion of a *weak* or *distributional* solution of the conservation laws (42). In words, we show that any solution that solves (42) on either side of a shock curve $(x(t), t)$ will be a weak solution if and only if the RH conditions (43) hold. To start then we must define the notion of weak solution. For this, we look to construct a condition that is equivalent to (42) for smooth solutions, but also applies to solutions that have jump discontinuities. The idea is to multiply the equation through by a smooth "test function" $\phi(x, t)$, integrate over any region, and integrate by parts to get the derivatives off the unknown functions u and $f(u)$, and onto the test function. The condition for a weak solution is then the condition that this integrated equation holds for all smooth test functions, and all three dimensional regions we integrate over. Now when we integrate by parts, there will be boundary terms, and to make these vanish, we assume the test function and all of its derivatives vanish outside the volume we integrate over. To make this precise, define:

Definition 9. *The support of a function $\phi(x, t)$, denoted $Supp\phi$, is the set of all values of (x, t) where the function ϕ is nonzero.*

Thus outside of any set that contains the support of ϕ, $\phi \equiv 0$, and so *phi* and all derivatives of ϕ vanish in the complement of that set. We use this to define a test function.

Definition 10. *A test function $\phi(x, t)$ is a smooth solution such that $Supp\phi$ is contained within a bounded set in (x, t) in $-\infty < x < \infty$, $t > 0$, that is some positive distance from $t = 0$.*

by *smooth* we mean that a test function ϕ can be differentiated any number of times, and the support condition is sufficient to guarantee that the function, together with all of its derivatives, vanish on the boundary of any set in $-\infty < x < \infty$, $t > 0$ that contains $Supp\phi$.

To get a condition for shock wave solutions $u(x,t)$, multiply equation (42) by a test function $\phi(x,t)$,

$$u_t\phi + f(u)_x\phi = 0,$$

integrate over (x,t)

$$\int_0^\infty \int_{-\infty}^\infty u_t\phi + f(u)_x\phi\, dxdt = \int\int_{\mathcal{K}} u_t\phi + f(u)_x\phi\, dxdt, \tag{44}$$

where we used the fact that $\phi(x,t)$ vanishes for (x,t) on the boundary of $\mathcal{K}$, because $\mathcal{K}$ contains the suppport of ϕ. Denoting the boundary of $\mathcal{K}$ by $\partial\mathcal{K}$ and assuming without loss of generality that $\mathcal{K}$ is contained within $-\infty < x < \infty$, $t > 0$, we can integrate (45) by parts to obtain

$$\begin{aligned} \int\int_{\mathcal{K}} (u_t\phi + f(u)_x\phi)\, dxdt &= \int\int_{\mathcal{K}} (u\phi)_t + (f\phi)_x\phi\, dxdt \\ &\quad - \int\int_{\mathcal{K}} u\phi_t + f(u)\phi_x\, dxdt. \end{aligned}$$

where we have applied the Leibniz product rule to write the integral as a divergence, plus the integration by parts term. By the divergence theorem, the first term reduces to an integral on the boundary $\partial\mathcal{K}$ where ϕ and all its derivatives vanish, so this term is zero, i.e.,

$$\begin{aligned} \int\int_{\mathcal{K}} (u_t\phi + f(u)_x\phi)\, dxdt &= \int\int_{\mathcal{K}} Div_{x,t}\,(f\phi, u\phi)\ dxdt \\ &\quad - \int_{\partial\mathcal{K}} (f\phi, u\phi)\cdot\mathbf{n}\, dxdt = 0. \end{aligned}$$

Thus we conclude that for any test function ϕ and solution u of (42) we have

$$\int_0^\infty \int_{-\infty}^\infty u_t\phi + f(u)_x\phi\, dxdt = -\int_0^\infty \int_{-\infty}^\infty u\phi_t + f(u)\phi_x\, dxdt, \tag{45}$$

so long as u is smooth enough so that the derivatives u_t and $f(u)_x$ exist. For *shock wave solutions*, the right hand side of (45) makes sense, but the left hand side of (45) does not.

Definition 11. *We call $u(x,t)$ a weak or distributional solution of the conservation law (42) if*

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + f(u)\phi_x\, dxdt = 0 \tag{46}$$

for all smooth test function $\phi(x,t)$.

The purpose of the section is to to use the Divergence Theorem to prove the following theorem which, in words, states that a shock wave solution is a weak solution of (42) if and only if the RH-jump conditions hold across the shock.

Theorem 12. *Assume $u(x,t)$ is a smooth solution of (42) on either side of a smooth shock curve $(x(t),t)$, but discontinuous across it. Then u is a weak solution if and only if $s[u]=[f]$ holds at each point of the shock. That is, if and only if*

$$\dot{x}(u_R-u_L)=(f(u_R)-f(u_L)).$$

Proof: Assume the ϕ is a smooth test function with support $\mathcal{K}$, and assume the values $u_L(x,t)$ of u on the left and $u_R(x,t)$ on the right of the shock curve $(x(t),t)$, both solve (42). Let $\mathcal{K}=\mathcal{K}_L\cup\mathcal{K}_R$ decompose $\mathcal{K}$ into the part left and right of the shock curve $(x(t),t)$, respectively, (c.f. Figure 1.) Thus we can write the weak condition as

$$\int_0^\infty\int_{-\infty}^\infty u\phi_t+f(u)\phi_x\,dxdt=\int\int_{\mathcal{K}}u\phi_t+f(u)\phi_x\,dxdt$$
$$=\int\int_{\mathcal{K_L}}u_L\phi_t+f(u_L)\phi_x\,dxdt+\int\int_{\mathcal{K_R}}u_R\phi_t+f(u_R)\phi_x\,dxdt,\quad(47)$$

where u_L and u_R are smooth in $\mathcal{K}$. Thus we can apply the divergence theorem in the derivation of the weak conditions *in reverse* to get the derivative back onto u and $f(u)$, and then apply (42). But the boundary condition at the shock, where ϕ need not vanish, will produce the RH condition. Starting on the left, we get

$$\int\int_{\mathcal{K_L}}u_L\phi_t+f(u_L)\phi_x\,dxdt=\int\int_{\mathcal{K_L}}(u_L\phi)_t+(f(u_L)\phi)_x\,dxdt$$
$$-\int\int_{\mathcal{K_L}}(u_L)_t\phi+f(u_L)_x\phi\,dxdt.$$

But by the divergence theorem,

$$\int\int_{\mathcal{K_L}}(u_L\phi)_t+(f(u_L)\phi)_x\,dxdt=\int_\Gamma\overrightarrow{(f_L,u_L)}\cdot\mathbf{n_L}\,\phi\,ds,\qquad(48)$$

and

$$\int\int_{\mathcal{K_L}}u_t\phi+f(u)_x\phi\,dxdt=0,$$

because u solves (42) in $\mathcal{K}_L$, so we obtain

$$\int\int_{\mathcal{K_L}}u_L\phi_t+f(u_L)\phi_x\,dxdt=\int_\Gamma\overrightarrow{(f_L,u_L)}\cdot\mathbf{n_L}\phi\,ds,$$

where Γ dentoes the shock curve $(x(t), t)$. Note that $u_t + f(u)_x$ makes sense and vanishes in $\mathcal{K}_{\mathcal{L}}$ and $\mathcal{K}_{\mathcal{R}}$ separately because we don't have to take derivatives at the shock itself. (We can take the derivatives all the way up to the shock curve by continuity of u and its derivatives on either side of the shock.) Note that the boundary of $\mathcal{K}_{\mathcal{L}}$ where ϕ need not vanish is exactly the shock curve $(x(t), t)$, which we have denoted by Γ, and $\mathbf{n}_L$ denotes the outer normal to $\mathcal{K}_L$.

Similarly on the right side $\mathcal{K}$ of the shock curve we obtain

$$\int\int_{\Gamma} u_R\phi_t + f(u_R)\phi_x \, dxdt = \int_{\Gamma} \overrightarrow{(f_R, u_R)} \cdot \mathbf{n_R} \, \phi \, ds.$$

Putting (49) and (49) together we obtain

$$\begin{aligned} &\int_{\Gamma} \overrightarrow{(f_L, u_L)} \cdot \mathbf{n_L}\phi \, ds + \int_{\Gamma} \overrightarrow{(f_R, u_R)} \cdot \mathbf{n_R} \, \phi \, ds \\ &\qquad = \int_{\Gamma} \overrightarrow{(f_R - f_L, u_R - u_L)} \cdot \mathbf{n_R} \, \phi \, ds = 0, \end{aligned} \tag{49}$$

where we used that $\mathbf{n}_L = -\mathbf{n}_R$. But on the shock curve $(x(t), t)$ the normal vector $\mathbf{n}_R = -\mathbf{i} + \dot{x}\mathbf{j}$. That is, the parametrization of the shock curve Γ with respect to t is

$$\mathbf{r}(t) = x(t)\mathbf{i} + t\mathbf{j},$$

so

$$\mathbf{v}'(t) = \dot{x}\mathbf{i} + \mathbf{j},$$

and the unit tangent vector $\mathbf{T} = |\mathbf{v}|/|\mathbf{v}|$ is

$$\mathbf{T} = \frac{\dot{x}\mathbf{i} + \mathbf{j}}{\sqrt{\dot{x}^2 + 1}}.$$

Hence the outer normal $\mathbf{n}_R$ to γ is

$$\mathbf{n}_R = \frac{-\mathbf{i} + \dot{x}\mathbf{j}}{\sqrt{\dot{x}^2 + 1}}.$$

Using this in (50) gives the result

$$\int_{\Gamma} \{s[u] + [f]\} \, \phi \, ds = 0,$$

The result, then, is that a weak solution $u(x, t)$ of the conservation law (42) that is a smooth solution on either side of a shock curve, must satisfy (50) for every smooth test function ϕ. It follows that we must have $s[u] + [f] = 0$ at each point of the shock, for if it were nonzero at some point on the shock curve, then we could cook up a test function with support near that point such that (50) was nonzero. Conversely, if $s[u] + [f] = 0$ all along the shock curve, and u is a strong solution on either side, then (50) is zero, and hence working backwards we would find that u is a weak solution as well. This completes the proof the theorem, and the derivation of the Rankine-Hugoniot jump conditions.