6–Energy Methods And The Energy of Waves
MATH 22C

1. Conservation of Energy

We discuss the principle of conservation of energy for ODE's, derive the energy associated with the harmonic oscillator, and then use this to guess the form of the continuum version of this energy for the linear wave equation. We then verify that this energy is conserved on solutions of the wave equation, and use it to solve the uniqueness problem for the wave equation. That is, we use the *energy of waves* to prove that the initial value problem for the wave equation has a unique solution always decomposable into the sum of left going and right going waves.

• Let's begin by asking the question: How much energy is stored in your car if it has mass m kg and is moving down the freeway at velocity v km per hour? The answer is

$$Kinetic\ Energy\ (\mathrm{KE}) = \frac{1}{2}mv^2.$$

That is, if you hit a wall and come to a complete stop at speed $v = 100\frac{km}{hr}$, and your car weighs $2000kg$, you will deliver

$$\begin{aligned} KE &= \frac{1}{2} \times 2000 \times 100^2 \frac{kg\ km^2}{hr^2} = 10^7 \times \frac{1000^2}{360^2}\frac{kg\ m^2}{s^2} \\ &= 7.7 \times 10^7 \mathrm{Newton-meter} \end{aligned}$$

of energy. One Newton-meter is the energy delivered to a mass of about $\frac{1}{10}kg$ falling one meter through Earth's gravity at the surface. In particular, that's how much energy the wall will deliver to you upon impact. It goes up by the *square* of the velocity, so that's why speeding is so dangerous.

So how do we know this? There are two main reasons. (1) It comes up in the expression for the work done by a force, and (2) It appears in the expression for the energy conserved by a *conservative force field*. When we treat the mass as a *point mass*, the framework is ODE's, when we view the mass as a continuous density, a mass per volume instead of a point mass, the framework is PDE's. In this section and the next, we discuss the point mass case (ODE's), and in the last section we treat the continuum version (PDE's) in the context of the wave equation.

(1) The argument for evaluating the work done by a force F on a mass m goes as follows:

$$The\ Work\ Done = W = Force \times Displacement.$$

So

$$\begin{aligned} W &= \int_a^b F \cdot ds = \int_a^b ma \cdot ds = \int_a^b m\frac{dv}{dt} \cdot ds \\ &= \int_{v_a}^{v_b} m\frac{ds}{dt} \cdot dv = \int_{v_a}^{v_b} mv \cdot dv \\ &= \frac{1}{2}mv^2]_{v_a}^{v_b} = \frac{1}{2}mv_a^2 - \frac{1}{2}mv_b^2 = \Delta KE. \end{aligned}$$

Thus the work done by a force is equal to the change in Kinetic Energy, so the work done by the wall as it moves through your car as you come to rest equals the KE=$\frac{1}{2}mv^2$ you started with.

(2) The second argument incorporates kinetic energy into the principle of conservation of energy. We say F is a conservative force if

$$F = -\frac{dU}{dy} = -U'(y)$$

for some function $U(y)$ called the potential energy. For example, let y denote distance from the Earth's surface, and assume a mass m moves vertically under the influence of gravity near the surface. Then Newton's law $F = ma$ gives the ODE,

$$m\ddot{y} = -mg.$$

To get the energy, we write the force $-mg$ as minus the derivative of the potential $U(y)$, that is,

$$m\ddot{y} = -\frac{d}{dy}\{mgy\}$$

so we have

$$U(y) = mgy$$

in this case. This is one instance to which the following very general principle applies.

Theorem 1. *If the force is minus the gradient of a potential $U(y)$, so that the ODE describing the motion is*

$$\ddot{y} = -U'(y),$$

then the energy

$$E = \frac{1}{2}m\dot{y}^2 + U(y)$$

is constant along each solution of the ODE.

This is so important that we give the

Proof: We show that $E(y(t))$ is constant along a solution $y(t)$ of the equation $\dot{y} = -U'(y)$. For this is suffices to show that $\frac{dE}{dt} = 0$. But by the chain rule we calculate:

$$\begin{aligned}\frac{d}{dt}E(y(t)) &= \frac{d}{dt}\left\{\frac{1}{2}\dot{y(t)}^2 + U(y(t))\right\} = \dot{y}\ddot{y} + U'(y)\dot{y} \\ &= \dot{y}\left\{\ddot{y}^2 + U(y)\right\} = 0\end{aligned}$$

because $y(t)$ solves the equation $\ddot{y}^2 + U'(y) = 0$.

Conclude: This is a mathematical theorem that holds for ODE's of form $\ddot{y} = -U'(y)$ independent of any interpretation of $\ddot{y}$ as a force: The function $E = \frac{1}{2}m\dot{y}^2 + U(y)$ is constant along solutions. The physical interpretation is that all along the motion the KE is the energy of motion, and it can be stored and released as potential energy, but the sum of the two is constant. Because we believe the fundamental forces of nature are conservative, fundamental forces must be conservative.

- As a first example consider the harmonic oscillator,

$$\ddot{y} + a^2 y = 0.$$

To put it in the form of conservative system and find the energy, solve for $\ddot{y}$ and write the "force" as a gradient:

$$\ddot{y} = -a^2 y = -\frac{d}{dy}\left\{a^2 \frac{1}{2} y^2\right\}.$$

Thus the potential energy is $U(y) = \frac{1}{2}a^2 y^2$, and the above theorem tells us that the total energy

$$E = \frac{1}{2}\dot{y}^2 + \frac{1}{2}a^2 y^2,$$

is constant along solutions of the harmonic oscillator.

We can immediately apply this to see that the trajectories of solutions in the $(y, \dot{y})$ plane must be ellipses. I.e., recall that $\ddot{y} + a^2 y = 0$ is equivalent to the first order system

$$\begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1}$$

with $u = y$ and $v = \dot{y}$. In (u, v)-variables, constant energy is

$$E = \frac{1}{2}v^2 + \frac{1}{2}a^2 u^2,$$

and this describes an ellipse in the (u, v)-plane. Thus the energy immediately tells us that solution of the harmonic oscillator must evolve on the constant energy curves which are ellipses. The energy by itself does not, however, tell us anything about how the solution moves around the ellipse in time.

• As a second *nonlinear* example, consider the nonlinear pendulum,

$$\ddot{y} + \frac{g}{L}\sin y = 0.$$

the equations which describe the swinging motion of a mass m at the end of a pendulum of length L, swinging about the equilibrium point $y = 0$ under the influence of gravity g. (Note the equation is independent of m!) Here $y = \theta$ is the angle the pendulum makes with the downward vertical rest point of the pendulum, so that $y = \theta = 0$ is the stable equilibrium at the bottom. (See Strogatz for a complete discussion and derivation.) Again, to put this in the form of a conservative system and find the energy, solve for $\ddot{y}$ and write the "force" as a gradient:

$$\ddot{y} = -\frac{g}{L}\sin y = -\frac{d}{dy}\left\{\frac{g}{L}(1-\cos y)\right\}.$$

Thus the potential energy is $U(y) = \frac{g}{L}(-\cos y)$, and the above theorem tells us that the total energy

$$E = \frac{1}{2}\dot{y}^2 + \frac{g}{L}(1-\cos y), \tag{2}$$

is constant along solutions of the harmonic oscillator. Note that we have normalized the anti-derivative as $1 - \cos y$ instead of $-\cos y$ in order that the minimum potential energy occur at $y = 0$, when the pendulum is at its lowest point. Expanding $\cos y = 1 - y^2 + O(1)y^4$ and letting $\frac{g}{l} = a^2$, we see that including the 1 also gives

$$U(y) = \frac{g}{L}(1-\cos y) = \frac{g}{L}y^2 + O(1)y^4$$

the correct constant so that it agrees with the potential energy $\frac{g}{L}y^2$ for the harmonic oscillator at the leading order. The main difference between the linear harmonic oscillator and the nonlinear pendulum is that the potential energy a^2y^2 is *unbounded* in the linear case, but *bounded* in the nonlinear case

$$0 \leq U(y) = \frac{g}{L}(1 - \cos y) \leq 2\frac{g}{L}.$$

Note also that, to give E the physical units of energy, say $kg\ m^2/s^2$, we would need only multiply (2) through by the mass mL^2. (We lost this factor when we divided through by the mass, and used the dimensionless angle θ as an unknown instead of the true distance along the moving pendulum.)

As in the case of the linear harmonic oscillator, we can again apply the formula for the energy to get information about the trajectories of solutions in the $(y, \dot{y})$ plane. Let's use it to see that there is a critical value of the energy above which the pendulum swings around with $\dot{y} = \dot{\theta}$ never changing sign, but below the critical energy, there must be a point where $\dot{y} = \dot{\theta}$ changes sign, the point where it starts *swinging the other way.*

For this, consider a fixed solution of the nonlinear pendulum, and write the energy (2) as

$$\frac{1}{2}\dot{y}^2 = E - \frac{g}{L}(1 - \cos y). \tag{3}$$

where E is constant along the solution. Now the potential energy is

$$U(y) = \frac{g}{L}(1 - \cos y),$$

which takes a maximum value of $2\frac{g}{L}$ when $\cos y = -1$. Thus there is a critical energy $E = 2\frac{g}{L}$, making for two distinct cases, according to whether $E > 2\frac{g}{L}$ or $E < 2\frac{g}{L}$.

If $E > 2\frac{g}{L}$, then the potential energy is never large enough to make the right hand side of (3) zero, in which case the left hand side, $1/2\dot{y}^2$ can never be zero along the solution, so the pendulum keeps swinging in the same angular direction, around and around.

On the other hand, if $E < 2\frac{g}{L}$, then there must be an angle $y(t)$ in the solution where the potential energy is equal to the energy E, i.e., $E = \frac{g}{L}(1 - \cos y(t))$. At this point, the KE must be zero, so the velocity $\dot{y}(t)$ must be zero at that point. That is, there must be a point where the pendulum comes to rest. But at this point we claim that the velocity must actually *change sign*, that is, the pendulum must change its direction of rotation. Indeed, if it did not, then the cosine would continue changing in the same direction at that point, making the potential energy larger than the energy E, the right hand side of (3) would go negative, making the KE$=\frac{\dot{y}^2}{2} < 0$ on the left hand side of (3) as well. But the KE is always positive because $\dot{y}^2 \geq 0$, so to avoid the contradiction, the velocity $\dot{y}$ must change sign at the point where $\dot{y} = 0$, as claimed.

Now a general principle of differential equations is that *the qualitative property of solutions can only change at critical values of dimensionless constants.* To find the dimensionless constant, note that $\dot{y}$ changes sign and the pendulum changes direction when

$$E = \frac{g}{L}(1 - \cos y(t)),$$

and it does not change direction when E exceeds the largest value on the right hand side. Dividing by $\frac{g}{L}$, our condition is that $\dot{y}$ changes sign when the dimensionless constant γ meets the condition

$$\gamma = \frac{L}{g}E = (1 - \cos y(t)).$$

The parameter γ is dimensionless because all terms in a physical equation must have the same dimensions, and $1 - \cos y$ on the right hand side is dimensionless. Thus the qualitative property of solutions changes at a value of the energy which makes γ equal to the largest value of $1 - \cos y$—above this value the pendulum has enough energy to swing over the top, but below this it swings back and forth. Conclude: The dimensionless constant is $\gamma = \frac{L}{g}E$, and its critical value $\gamma = \gamma_*$ is $\gamma_* = 2$.

2. Energy Conservation for Point Masses (ODE's) in Three Dimensions

The physical principle of conservation as expressed in (1) and (2) above can be generalized to more realistic motion in three dimensions by introducing line integrals. This is a topic of *Vector Calculus*. In this section we describe the mathematical theory and its connection to the physics of motion for multiple interacting point masses described by *systems* of ODE's. This is very beautiful mathematics, and completes the ODE picture of conservation of energy for point masses. In the next section we extend the notion of conservation to PDE's in the context of the wave equation.

To generalize (1) to three dimensions, we first need to define the work done by a force field, and for this we must recall the line integral. So let $\mathbf{F}(x,y,z)$ denote a vector field in $\mathcal{R}^3$ which we interpret as a force field,

$$\mathbf{F}(x,y,z) = F_1(xy,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k},$$

where (F_1, F_2, F_3) are the components of the force at (x,y,z). Then given a curve

$$\mathcal{C}:\ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

$a \le t \le b$, the *work* done by $\mathbf{F}$ on a point mass m as it moves along $\mathcal{C}$ is defined as

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds, \tag{4}$$

which is called the line integral of $\mathbf{F}$ along $\mathcal{C}$. Recall that the line integral is defined as the limit of the Riemann sum

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{T}_i \, \Delta s, \tag{5}$$

where

$$\mathbf{F}_i = \mathbf{F}(\mathbf{r}(t_i))$$

is the force,

$$\mathbf{T}_i = \frac{\mathbf{r}'(t_i)}{\|\mathbf{r}'(t_i)\|}$$

is the unit tangent vector on $\mathcal{C}$ at time t_i, ds is arc length along $\mathcal{C}$ because

$$\frac{ds}{dt} = \|\mathbf{v}(t)\|, \quad \mathbf{v}(t) = \mathbf{r}'(t),$$

$\mathbf{v}(t)$ is the velocity vector tangent to $\mathcal{C}$ at $\mathbf{r}(t)$, and as usual, t_i denote the mesh points obtained by dividing $[a, b]$ into N equal intervals of size

$$\Delta t = \frac{|b-a|}{N}, \quad t_0 = a, \quad t_i = a + i\Delta t, \quad t_N = b.$$

Thus the line integral is the natural generalization of work because it is the limit of the sum of force times displacement along the curve. Recall that by the substitution principle, it has the nice property that it can be evaluated in terms of any parameterization $\mathbf{r}(t)$ of the curve $\mathcal{C}$ by

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt, \tag{6}$$

thereby reducing the line integral for work to an elementary integral of first quarter calculus.

Things start to get very interesting, as we saw in Vector Calculus, when we apply these principles to the case when the force is **conservative**. We say $\mathbf{F}$ is *conservative* if

$$F(\mathbf{y}) = -\nabla f(\mathbf{y}),$$

for some scalar function $f(\mathbf{y})$ called the *potential.* The following mathematical theory generalizes the fundamental theorem of calculus to line integrals:

Theorem 2. *Assume F is conservative, so $F(\mathbf{y}) = -\nabla f(\mathbf{y})$ for a scalar function f. Then*

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}ds = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{7}$$

Proof: This follows directly from the definition of line integral as follows:

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}ds &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt \qquad (8)\\ &= \int_a^b \frac{d}{dt}\mathbf{F}(\mathbf{r}(t))dt\\ &= \mathbf{F}(\mathbf{r}(b)) - \mathbf{F}(\mathbf{r}(a)),\end{aligned}$$

where we have applied the chain rule and then the regular Fundamental Theorem of Calculus.

Comment: The definition of the line inertial in (6) is purely mathematical, but its meaning as work comes to light when we impose the vector form of Newton's second law $\mathbf{F} = m\mathbf{a}$. Two problems then immediately come to mind, one of them easy and the other one hard. The easy problem is this: Given a curve $\mathbf{r}(t)$, find the the total force that is creating the motion. This is easy if we have formulas for the components of $\mathbf{r}(t) = (x(t), y(t), z(t))$ because we can then just calculate $\mathbf{a} = \mathbf{r}''(t) = (x''(t), y''(t), z''(t))$ and solve for the force $\mathbf{F} = m\mathbf{a}$. The hard problem is: Given the force $\mathbf{F}$, find the trajectory of the particle $\mathbf{r}(t)$

that gives the motion created by that force. Then $\mathbf{F} = m\mathbf{a}$ gives you an equation to solve for, namely, $\mathbf{r}''(t) = \frac{1}{m}\mathbf{F}$. In particular, if the force depends on the position, this can be a complicated ODE to solve for the position $\mathbf{r}(t)$, and initial conditions will be required. To keep the following arguments clear, it is important to keep in mind that many forces may be acting on a mass m to create the motion $\mathbf{r}(t)$, and for each separate force we can define the line integral of the work done by that force, but it is only the sum of all the forces, the *total force*, that creates the total acceleration through $\mathbf{F} = m\mathbf{a}$.

The final justification of (6) as the correct generalization of work to three space dimensions comes from the following generalization of (1) and (2). In (1) we prove that the work done is equal to the change in kinetic energy. In (2) we prove that Kinetic Energy+Potential Energy is conserved when the total force creating the motion is *conservative.* For this we must assume that $\mathbf{F}$ is the TOTAL force which is accelerating the mass along the curve $\mathcal{C}$.

(1) Assume $\mathbf{r}(t)$ is the trajectory of a point mass m moving according to Newton's law $\mathbf{F} = m\mathbf{a}$, meaning implicitly that $\mathbf{F}$ is the sum of all the forces creating the acceleration, and therefore $\mathbf{a}(t) = \mathbf{r}''(t)$ is the vector acceleration of the particle determined by its trajectory $\mathbf{r}(t)$. Then

$$
\begin{aligned}
W &= \int_a^b \mathbf{F}\cdot\mathbf{T}ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt \\
&= \int_a^b m\,\mathbf{a}\cdot\mathbf{v}\,dt = \int_a^b m\,\mathbf{r}''(t)\cdot\mathbf{r}'(t)\,dt \\
&= \int_a^b \frac{1}{2}m\frac{d}{dt}\left\{\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right\}\,dt = \int_a^b \frac{1}{2}m\frac{d}{dt}\left\{\|\mathbf{v}(t)\|^2\right\}\,dt \\
&= \frac{1}{2}m\|\mathbf{v}(b)\|^2 - \frac{1}{2}m\|\mathbf{v}(a)\|^2 = \Delta KE.
\end{aligned}
$$

Thus the work done by a force is equal to the change in Kinetic Energy for general motions in three dimensions!

(2) The general principle of conservation of energy now holds when the total force $\mathbf{F}$ acting on m is *conservative.* I.e., assume

$$\mathbf{F} = -\nabla U$$

for some scalar function $U(\mathbf{y})$ called the potential energy. Applying (7) with $f = -U$ we conclude the work done by $\mathbf{F}$ along $\mathcal{C}$ is not just the change in kinetic energy ΔKE (as shown in (1)), but also, but Theorem 2, is equal to minus the change in potential energy ΔPE as well, namely, we have:

$$\begin{aligned} W &= \int_a^b \mathbf{F} \cdot \mathbf{T} ds \\ &= -U(\mathbf{r}(b)) + U(\mathbf{r}(a)) = -\Delta PE. \end{aligned} \tag{9}$$

Since by (1) W equals the change in KE, and by (9) it equals minus the change in PE, so $\Delta KE + \Delta PE = 0$ all along the curve, we must have

$$E = KE + PE = \frac{1}{2}m\|\mathbf{v}(t)\|^2 + U(\mathbf{r}(t)) = const$$

all along the curve. Conclude: Energy is conserved in a conservative force field.

Another way to express conservation of energy is in terms of ODE's, and this is the expression used in differential equations. For this, notice that Newton's law $F = ma$ for a conservative force field can be viewed as a system of equations for the curve $\mathbf{y}(t)$, (we use $\mathbf{y}$ in place of $\mathbf{r}$ for ODE's), namely

$$m\ddot{\mathbf{y}}''(t) = -\nabla U(\mathbf{y}), \tag{10}$$

where $\mathbf{y} = (x, y, z)$. Conservation of energy can then be re-expressed as the following theorem about equations of form (10).

Theorem 3. *The quantity*

$$E = \frac{1}{2}m\|\dot{\mathbf{y}}\|^2 + U(\mathbf{y}),$$

which we call the sum of the kinetic and potential energy, is always constant along solutions of (10).

We have already proven this through the notion of the work done W which was expressed two ways, as a change in KE and a change in PE. But we can prove this directly without appealing to the notion of work at all:

Proof: Assume $\mathbf{y} = \mathbf{r}(t)$ is a solution of (10), so $\dot{\mathbf{y}} = \mathbf{v}(t)$. Then for this solution, the energy is

$$E(t) = \frac{1}{2}m\|\mathbf{v}(t)\|^2 + U(\mathbf{r}(t)).$$

To prove $E(t)$ is constant along the solution $\mathbf{r}(t)$, we prove $\dot{E} = 0$ for every t. Writing $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$ and taking the derivative using the product rule for the dot product together with the chain rule, we obtain

$$\begin{aligned} \dot{E} &= \frac{d}{dt}\left\{\frac{1}{2}m\left(\mathbf{v}(t)\cdot\mathbf{v}(t)\right)^2 + U(\mathbf{r}(t))\right\} \qquad (11) \\ &= m\mathbf{v}'(t)\cdot\mathbf{v}(t) + \nabla U\cdot\mathbf{r}'(t) \\ &= m\mathbf{a}(t)\cdot\mathbf{v}(t) + \nabla U\cdot\mathbf{v}(t) \\ &= -\nabla\mathbf{U}\cdot\mathbf{v}(t) + \nabla U\cdot\mathbf{v}(t) = 0. \end{aligned}$$

We used the product rule

$$\frac{d}{dt}\left(\mathbf{v}(t)\cdot\mathbf{v}(t)\right)^2 = \mathbf{v}'(t)\cdot\mathbf{v}(t) + \mathbf{v}(t)\cdot\mathbf{v}'(t) = 2\mathbf{a}(t)\cdot\mathbf{v}(t),$$

and the chain rule

$$\begin{aligned} \frac{d}{dt}U(\mathbf{r}(t)) &= \frac{d}{dt}U(x(t), y(t), z(t)) \\ &= \frac{\partial U}{\partial x}x'(t) + \frac{\partial U}{\partial y}y'(t) + \frac{\partial U}{\partial z}z'(t) \\ &= \nabla U\cdot\mathbf{r}'(t). \end{aligned}$$

Thus the energy is constant because when the force is conservative, the derivative of the kinetic energy is minus the derivative of the potential energy.

In fact, this mathematical expression of conservation extends immediately to equations with any number of unknowns $\mathbf{y} = y_1, ..., y_n)$. For example, it extends to the motion of $n = 10^{23}$ molecules in a container. The generalization goes as follows. We say $\mathbf{F}$ is conservative if $\mathbf{F} = -\nabla U$ for some scalar function U. Then the equations are (we take $m = 1$)

$$\ddot{\mathbf{y}} = -\nabla U(\mathbf{y}),$$

and the theorem says that energy is conserved in the sense that the energy

$$E = \frac{1}{2}\|\dot{\mathbf{y}}\|^2 + U(\mathbf{y}),$$

is constant along solutions. The proofs of (1) and (2) are identical for general n as in the case $n = 3$. Just retain the vector form of the equations in each step, and interpret $\mathbf{y}$ as being in $\mathcal{R}^n$ instead of $\mathbf{R}^3$. This is a very general mathematical principle.

3. The Energy of Waves

Recall that the initial value problem for the wave equation is

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \\ u(x,0) &= h(x), \\ u_t(x,0) &= k(x), \end{aligned} \tag{12}$$

a problem that asks for a solution $u(x,t)$ given initial data functions h and k. That is, the wave equation is second order in time, so by analogy with the second order harmonic oscillator, we should be able to assign the *initial value* $u(x,0)$ and the *initial velocity* $u_t(x,0)$. Now in Section 4 we found a solution to the initial value problem (12) in terms of a left going wave and a right going wave. The result was that one particular solution of (12) is given by

$$u(x,t) = f(x+ct) + g(x-ct), \tag{13}$$

where

$$\begin{aligned} f(x) &= \frac{1}{2}\int_0^x \left[h'(\xi) + \frac{1}{c}k(\xi)\right] d\xi + f(0), \\ g(x) &= \frac{1}{2}\int_0^x \left[h'(\xi) - \frac{1}{c}k(\xi)\right] d\xi + g(0), \end{aligned} \tag{14}$$

with the requirement

$$f(0) + g(0) = h(0).$$

This then defines the functions f and g in terms of the given h and k so that (13) solves (12).

But although (14) is one solution, how do we know there aren't some other solutions hanging around? Maybe solutions that cannot be decomposed into left and right going waves? The answer is settled once we prove that for a given h and k, the initial value problem (12) has one and only one *unique* solution. This is what we can prove by using the *energy*.

So to this end, we ask, what is the energy contained in a solution $u(x,t)$ of the wave equation? The answer is that at each point, the kinetic energy is

$$KE = \frac{1}{2}u_t^2 \ ,$$

(not too hard to guess), and the potential energy is

$$PE = \frac{1}{2}c^2u_x^2 \ ,$$

(not so easy to guess, but not unreasonable). But given the energy is KE+PE at each point, the total energy at a given time t should be the sum of all the kinetic plus potential energies, summed over all the points at time t. But this is calculus, and this sum only makes sense as an integral. Here is the energy of a solution $u(x,t)$ at a given time t:

$$E(t) = \int_{-\infty}^{+\infty} \frac{1}{2}u_t^2 + \frac{1}{2}c^2u_x^2 \; dx.$$

But the only real proof that this is the correct energy of the wave $u(x,t)$ is the following theorem, which states that the energy so defined, is *constant* in time along any solution of the wave equation.

Theorem 4. *If $u(x,t)$ solves the wave equation,*

$$u_{tt} - c^2u_{xx} = 0,$$

then $E(t)$ is constant in time. That is,

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{-\infty}^{+\infty} \frac{1}{2}u_t^2(x,t) + \frac{1}{2}c^2u_x^2(x,t) \; dx = 0. \quad (15)$$

Before proving the Theorem, we first demonstrate its utility by using it to prove the uniqueness of solutions of the initial value problem (12).

•**Uniqueness:** Assuming Theorem 4, we can prove:

Theorem 5. *Assume $u(x,t)$ and $v(x,t)$ are smooth functions of finite energy which both solve the initial value problem (12) for the same given functions $h(x)$ and $k(x)$. Then*

$$u(x,t) = v(x,t)$$

for all $x \in \mathcal{R}$, $t \geq 0$.

Proof: Assume then that both $u(x,t)$ and $v(x,t)$ solve (12). We prove that $w(x,t) = u(x,t) - v(x,t) = 0$ for all (x,t). Since the wave equation $u_{tt} - c^2u_{xx} = 0$ is linear and homogeneous, superposition holds, and so we know that $w = u - v$ also solves the wave equation. That is

$$(u-v)_{tt} - c^2(u-v)_{xx} = u_{tt} - c^2 u_{xx} + v_{tt} - c^2 v_{xx} = 0.$$

But u and v satisfy the same initial conditions,

$$\begin{aligned} u(x,0) - v(x,0) &= h(x) - h(x) = 0, \\ u_t(x,0) - v_t(x,0) &= k(x) - k(x) = 0. \end{aligned}$$

That is, the function

$$w(x,t) = u(x,t) - v(x,t),$$

solves the wave equation with zero initial data,

$$w(x,0) = 0 = w_t(x,0).$$

Thus by Theorem 4, the energy is constant in time along the solution $w(x,t)$ and the energy starts out zero, so

$$E(t) = E(0) = \int_{-\infty}^{+\infty} \frac{1}{2} w_t^2(x,0) + \frac{1}{2} c^2 w_x^2(x,0)\ dx = 0,$$

because w and w_t are both zero at $t = 0$. Therefore, at each time $t \geq 0$ we have

$$E(t) = \int_{-\infty}^{+\infty} \frac{1}{2} w_t^2(x,t) + \frac{1}{2} c^2 w_x^2(x,t)\ dx = 0. \qquad (16)$$

But the integrand in the integral (16) is always non-negative,

$$\frac{1}{2} w_t^2(x,t) + \frac{1}{2} c^2 w_x^2(x,t) \geq 0,$$

and the only way an integral over a nonnegative function can be zero is if the integrand is zero,

$$\frac{1}{2} w_t^2(x,t) + \frac{1}{2} c^2 w_x^2(x,t) \geq 0.$$

That is, the only way the area under the graph of a non-negative function can be zero is if the function is zero. It follows then that both

$$w_t = 0, \quad w_x = 0$$

for all x and $t \geq 0$. But w_t and w_x are the two components of the gradient of w, so we have

$$\nabla w\,(x,t) = 0.$$

Now the only way the gradient of a function can vanish is if the function is constant, so we must have

$$w(x,t) = const.$$

And since $w(x,0) = 0$, it follows that the constant must be zero,

$$w(x,t) = u(x,t) - v(x,t) = 0.$$

We conclude that the two solutions u and v that solve the same initial value problem must be equal, and therefore there can only be one unique solution to the initial value problem, as claimed. □

• To finish the section, we now give the proof of Theorem 4. The main technique we will need is integration by parts for partial derivatives. It's really no different than regular integration by parts so long as you ignore the other variables hanging around in the expression. For integration by parts, assume you have two functions $u(x)$ and $v(x)$, and use the PDE notation $v_x = v'(x)$. Then by the product rule,

$$(uv)_x = u_x v + u v_x,$$

so by the Fundamental Theorem of Calculus

$$\int_{-\infty}^{+\infty} u v_x \, dx = \int_{-\infty}^{+\infty} (uv)_x - u_x v \, dx = uv\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u_x v dx.$$

In particular, if we assume the functions u and v vanish at $x = \pm\infty$, then

$$\int_{-\infty}^{+\infty} u v_x \, dx = \int_{-\infty}^{+\infty} (uv)_x - u_x v \, dx = -\int_{-\infty}^{+\infty} u_x v dx.$$

Note that the formula would be no different if u and v depended on x and t, so long as

$$\lim_{x\to\pm\infty} u(x,t) = 0 = \lim_{x\to\pm\infty} v(x,t),$$

namely

$$\int_{-\infty}^{+\infty} u(x,t)v_x(x,t)\,dx = -\int_{-\infty}^{+\infty} u_x(x,t)v(x,t)\,dx.$$

All the action is going on in the variable x that is being integrated, and the t just goes along for the ride.

The final thing we need in order to see that the energy is constant along solutions of the wave equation, is a formula for differentiating through an integral sign. That is, consider the general problem of differentiating $G(t)$ with respect to t when

$$G(t) = \int_{\infty}^{\infty} f(x,t)dx.$$

We want to see that

$$G'(t) = \frac{d}{dt}\int_{\infty}^{\infty} f(x,t)dx = \int_{\infty}^{\infty} \frac{\partial}{\partial t} f(x,t)dx.$$

We can see this directly by writing the derivative as a limit, and using properties of limits to pass the limits through the integral sign. That is,

$$\begin{aligned}
\frac{d}{dt}\int_{\infty}^{\infty} f(x,t)dx &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int_{\infty}^{\infty} f(x,t+\Delta t)dx - \int_{\infty}^{\infty} f(x,t)dx\right\} \\
&= \lim_{\Delta t\to 0}\left\{\frac{\int_{\infty}^{\infty} f(x,t+\Delta t)dx - \int_{\infty}^{\infty} f(x,t)dx}{\Delta t}\right\} \\
&= \lim_{\Delta t\to 0}\int_{\infty}^{\infty}\frac{f(x,t+\Delta t) - f(x,t)dx}{\Delta t} \\
&= \int_{\infty}^{\infty}\frac{\partial}{\partial t} f(x,t)dx.
\end{aligned}$$

The point is that to justify all the steps, you only need the differentiability of f and that $\lim_{x\to\pm\infty} f(x,t) = 0$ rapidly enough that all the integrals and sums are uniformly finite. (Getting the precise conditions straight is a topic in Advanced Calculus.)

So, to prove the energy $E(t)$ is constant along solutions, differentiate (15),

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{d}{dt}\int_{-\infty}^{+\infty} \frac{1}{2}u_t^2(x,t) + \frac{1}{2}c^2 u_x^2(x,t)\ dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{2}\frac{\partial}{\partial t}u_t^2(x,t) + \frac{1}{2}c^2\frac{\partial}{\partial t}u_x^2(x,t)\ dx \\
&= \int_{-\infty}^{+\infty} u_t u_{tt}(x,t) + c^2 u_x(x,t)u_{xt}(x,t)\ dx \\
&= \int_{-\infty}^{+\infty} u_t u_{tt}(x,t) - c^2 u_{xx}(x,t)u_t(x,t)\ dx \\
&= \int_{-\infty}^{+\infty} u_t \left\{u_{tt}(x,t) - c^2 u_{xx}(x,t)\right\}\ dx = 0,
\end{aligned}$$

where we have first differentiated through the integral sign, then applied the chain rule, and finally integration by parts on the last term. The final expression has the wave equation as a factor in the integrand, which therefore vanishes because u is assumed to be a solution of the wave equation at the start. We conclude that if the energy is finite, then it is constant on solutions of the wave equation, so this completes the proof of Theorem 4. □

Conclude: We have shown that the energy

$$E = \int_{-\infty}^{+\infty} \frac{1}{2}u_t^2(x,t) + \frac{1}{2}c^2 u_x^2(x,t)\ dx$$

is constant along solutions of the wave equation. Using this, together with the linearity of solutions, we prove that the difference between two solutions stays zero if it starts out zero, thereby proving *uniqueness* of the initial value problem. It follows that the solution of the initial value problem we constructed as the superposition of a left going wave and a right going wave, is the only solution, (at least among those of finite energy!)