9–Shock Waves and the Riemann Problem

MATH 22C

1. Introduction

The purpose of this section is to solve the so called Riemann problem for Burgers equation and for the p-system. The Riemann problem is the initial value problem when the initial data consists of two constant states U_L and U_R separated by a jump discontinuity at $x = 0$. That is, the initial value problem

$$U_t + f(U)_x = 0, \tag{1}$$

$$U(x, 0) = U_0(x), \tag{2}$$

where

$$U_0(x) = \begin{cases} U_L, & \text{for } \; x \leq 0, \\ U_R, & \text{for } \; x \geq 0. \end{cases}$$

Shock wave theory only applies to equations in conservation form (10), in which a total derivative falls on the nonlinear function f. For *systems* of conservation laws like the compressible Euler equations, $U = (U_1, ..., U_n)$ and $f(U) = (f_1(U), ..., f_n(U))$, and the components U_i of U are called the *conserved quantities* and the components $f_i(U)$ of $f(U)$ are called the *fluxes*. For example, the compressible Euler equations take the conservation form

$$\rho_t + G_x = 0, \tag{3}$$

$$G_t + \left(G^2/\rho + p\right)_x = 0, \tag{4}$$

with conserved quantities $U = (\rho, G)$, $G = \rho u$, and fluxes $f(U) = (G, G^2/\rho + p(\rho))$.

In the case of the compressible Euler equations, and its equivalent formulation as the p-system, the Riemann problem poses the shock tube problem, the problem when the density and velocity of a gas at time zero are constant states separated by a membrane. When the membrane is removed, waves move in both directions down the shock tube, and the Riemann problem determines exactly what the waves will be. The predictions agree with experiment. The Riemann problem is a building block for more general solutions of conservation laws, as well as for numerical schemes to numerically simulate solutions. The simple-waves of the previous section provide solutions of the Riemann problem when the waves are expansive. Such *centered simple waves* are called *rarefaction waves*. But when solutions are compressive, shock waves are required to complete the picture.

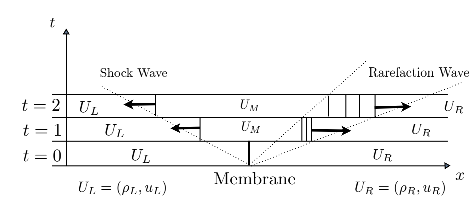

Figure 1a: Waves in a Shock Tube

To describe the rarefaction waves in terms of simple waves, recall the *Simple-Wave Principle* of the previous section stated that simple wave solutions of

$$U_t + A(U)U_x = 0 \tag{5}$$

can be constructed by imposing the condition that states on the integral curve $\mathbf{R}(U_L)$ propagate at speed given by the eigenvalue at that state. Here $\mathbf{R}(U_L)$ denotes the integral curve of the eigenvector $R = R(U)$ from the eigenfamily $(\lambda, R) \equiv (\lambda(U), R(U))$, passing through the state U_L. Thus, if the eigenvalue increases from a state U_L to $U_R \in \mathbf{R}(U_L)$, then the Riemann problem can be solved by asking that each state $U \in \mathbf{R}(U_L)$ between U_L and U_R propagate at speed $\lambda(U)$. This creates a rarefaction wave connecting U_L to U_R by a wave in the (x,t)-plane. (See Figures 1 and 2.) But such waves only make sense when λ *increases* from U_L to U_R along $\mathbf{R}(U_L)$, and cannot be used to create waves between states when λ decreases. That is, when the λ-eigenfield is genuinely nonlinear GN,

$$\nabla\lambda \cdot R \neq 0,$$

then λ is monotone along all the integral curves $\mathbf{R}$ and can be taken as the parameter along each one. In this case, for a given left state U_L, the λ-rarefaction waves can solve the Riemann problem for all $U_R \in \mathbf{R}^+(U_L)$, where

$$\mathbf{R}^+(U_L) = \{U_R \in \mathbf{R} : \lambda(U_R) > \lambda(U_L)\}.$$

That is, for GN fields, $\mathbf{R}^+(U_L)$ is the half of the integral curve $\mathbf{R}(U_L)$ along which λ increases from U_L. Shock waves are required to extend the rarefaction waves to a complete solution of the Riemann problem. Our goal is to show that for each state U_L there is a shock curve $\mathbf{S}^-(U_L)$ tangent to the integral curve $\mathbf{R}^+(U_L)$ at U_L, with matching second derivative as well, (we say they make C^2 contact at U_L), such that the concatenation

$$W(U_L) = \mathbf{R}^+(U_L) \cup \mathbf{S}^-(U_L)$$

creates a wave curve along which the Riemann problem can be solved whether λ increases or decreases from U_L to U_R along $W(U_L)$. (See Figures 1 and 3.) We will show that each of the characteristic families of the p-system completes to such a wave curve $W_i(U_L)$, $i = 1, 2$, and the concatenation of these wave curves provides a coordinate system in U-space centered at U_L that tells how to solve the Riemann problem for every U_R. The solution is unique within the the class of admissible shock waves and rarefaction waves.

An important point to make is that the theory of shock waves only applies to equations in conservation form,

$$U_t + f(U)_x = 0, \tag{6}$$

in which a total derivative falls on the nonlinear function f, c.f. (10)). But the *simple-wave* form of the equations (5), not the conservation form (6), is required to describe the simple-waves. To get the simple wave form of the equations from the conservation form we must differentiate the flux f with respect to U and then U with respect to x, That is,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial U}\frac{\partial U}{\partial x} = A(U)U_x.$$

For a system of conservation laws in which $U = (U_1, ..., U_n)$ and $f(U) = (f_1(U), ..., f_n(U))$, the matrix $A(U)$ is denoted df, and is simply the matrix obtained by putting the gradient $\nabla f_i(U)$ in the $i'th$ row to create an $n \times n$ matrix of derivatives of f. Below we will use the conservation form of the equations to determine the shock waves, and the simple-wave form of the equations to determine the rarefaction waves, for Burgers equation and the p-system.

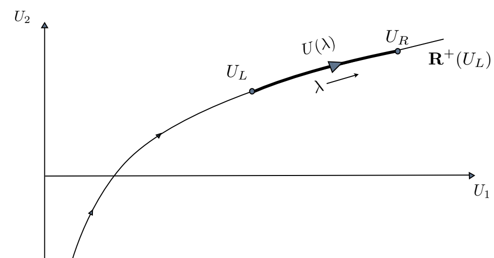

Figure 1: States on a λ-simple wave

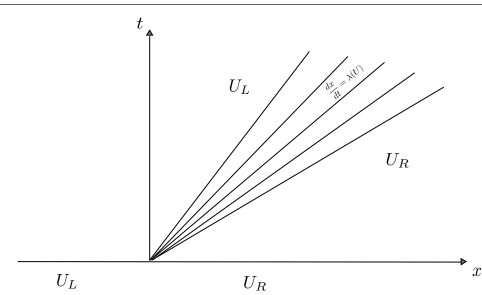

Figure 2: Centered Simple-Wave=Rarefaction Wave

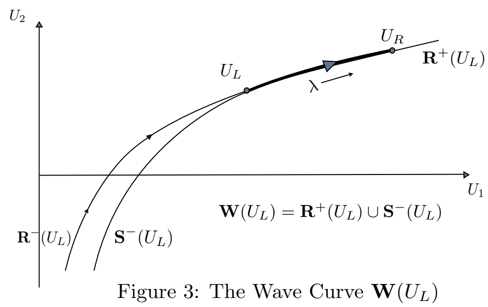

Figure 3: The Wave Curve $\mathbf{W}(U_L)$

2. The Rankine-Hugoniot Jump Condition for Shock Waves

The principle for describing shock waves is the Rankine-Hugoniot (RH) jump condition. The RH condition applies to solutions $U(x,t)$ of equations in conservation form

$$U_t + f(U)_x = 0, \tag{7}$$

when $U(x,t)$ is smooth on either side of a smooth curve $(x(t),t)$ in the (x,t)-plane, but jumps from $U_L(t)$ to $U_R(t)$ across it. (See Figure 4.) In particular, when U_L and U_R are constant, RH implies the speed s is constant as well. We will derive (7) as one of the applications of the divergence theorem in the next section, but for now we accept it. The Rankine-Hugoniot jump condition states that to be a legitimate shock wave, the speed of the shock $s = \dot{x}$ must, at every time t, be related to the jump in u across the shock by the relation

$$s[U] = [f(U)]. \tag{8}$$

The standard notation here is that brackets around a quantity $[\cdot]$ indicate the jump in the quantity from left to right across the shock, (left and right measured in the (x,t)-plane), so that $[U] = U_R - U_L$ and $[f(U)] = f(U_R) - f(U_L)$. (See Figure 4.)

For example, the conservation form of the Burgers equation is

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0,$$

with flux function

$$f(u) = \frac{1}{2}u^2.$$

In this case the role of RH is to give the speed of a shock in terms of the right and left states. That is, solving for the speed s in RH for Burgers equation gives an expression for the shock speed in terms of the right and left states across a Burgers shock:

$$s = \frac{[f]}{[u]} = \frac{1}{2}\frac{u_R^2 - u_L^2}{u_L - u_R} = \frac{u_L + u_R}{2}.$$

Conclude that for the scalar Burgers equation, the RH conditions simply tell us that the speed of a legitimate shock must be the average of the states on the left and right of the shock. For system of equations, unraveling the meaning of the RH condition (8) is more problematic. A number of comments are in order.

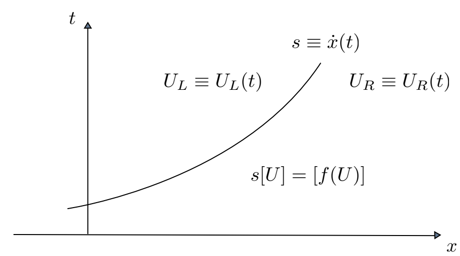

Figure 4: The Rankine-Hugoniot Jump Condition for Shocks

• Note that the fundamental starting point of the theory of shock waves is the conservation form of the equations (7). The RH jump condition (8) only makes sense for equations in conservation form. It tells the quantities that are conserved, and these determine conservation across the shock waves. As an example, multiplying Burgers through by u gives

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0,$$

which has the same simple-wave form

$$u_t + uu_x = 0,$$

but the conserved quantity is u^2 not u, so the RH condition is not the same condition. Thus, the conservation form of the equations is determined by the physically meaningful quantities that are conserved. For example, the conservation form of the compressible Euler equations is

$$\begin{aligned} \rho_t + G_x &= 0, \\ G_t + (G^2/\rho + p)_x &= 0, \end{aligned} \tag{9}$$

of form

$$U_t + f(U)_x = 0,$$

(upper case U to distinguish from the velocity u!), with $U = (\rho, G)$ and $f(U) = (G, G^2/\rho + p)$. Since mass and momentum are what is conserved across a shock wave, and the conserved quantities are the mass density ρ and momentum density $G = \rho u$, these are the physically correct conserved quantities and hence this is the physically correct conservation form of the equations.

• Finally, note that if states $U_L = U_1$ and $U_R = U_2$ meet the RH conditions

$$s[U] = [f(U)],$$

then minus-ing the jumps on both sides shows $U_L = U_2$ and $U_R = U_1$ does also...the RH conditions can't distinguish between U_L on the left and U_R on the right, and the reverse. But in fact, only one of these will produce a stable shock wave. A large part of the theory of shock waves involves the study of *entropy conditions* that rule out the unstable shocks that have U_L and U_R on the *wrong side*. We will presently see how to do this for the Burgers equation and for the p-system and compressible Euler equations of gas dynamics.

3. The Riemann Problem for Burgers Equation

The Riemann problem is the initial value problem when the initial data consists of two constant states u_L and u_R separated by a jump discontinuity at $x = 0$. (We use lower case u for the unknown because it is a scalar.) That is, the initial value problem

$$u_t + f(u)_x = 0, \tag{10}$$
$$u(x,0) = u_0(x), \tag{11}$$

where

$$u_0(x) = \begin{cases} u_L, & \text{for} \quad x \le 0, \\ u_R, & \text{for} \quad x \ge 0. \end{cases}$$

First, we can construct the rarefaction wave solutions of the Riemann problem when $u_L < u_R$ from the Simple-Wave Principle. To obtain the simple wave form of the equations from the conservation form, differentiate the flux. That is, for a general scalar conservation law

$$u_t + f(u)_x = 0,$$

differentiate f by the chain rule to obtain

$$f(u)_x = f'(u)u_x.$$

Thus the simple wave principle says $\lambda = f'(u)$, $R = 1$, and simple waves are constructed by asking that solutions $u(x,t)$ be constant along lines of speed $\dot{x} = \lambda$. In the case of Burgers equation,

$$f(u) = \frac{1}{2}u^2,$$

and so the simple-wave form of Burgers equation is

$$u_t + uu_x = 0.$$

Thus Burgers equation expresses that $u(x,t)$ should be *constant along lines of speed* $\dot{x} = u$, and so for Burgers we can construct the rarefaction wave solutions of the Riemann problem when $u_L < u_R$ by asking that each value of u between u_L and u_R should propagate at speed u: Such a solution creates a wave fan, or rarefaction wave, taking u_L to u_R. When $u_L > u_R$, the rarefaction waves don't exist because the speeds decrease from left to right, and the fan of values expands inconsistently back toward u_L. In this case we take the shock wave solution that takes u_L to u_R. As above, according to the RH condition, the speed of the shock is the average of the speeds on each side,

$$s = \frac{[f]}{[u]} = \frac{u_L + u_R}{2}.$$

Taking rarefaction waves when $u_L < u_R$ and shock waves when $u_L > u_R$ solves the Riemann problem for every u_L, u_R. The solution is pictured most easily in on the graph of f as a function of u, (see Figure 5). By the simple-wave formulation,

$$u_t + f'(u)u_x = 0,$$

the speed of a wave is $\lambda = f'(u)$, which for Burgers happens to be $f'(u) = u$. Thus the rarefaction wave speeds for states u between u_L and $u_R > u_L$ are the slopes of the tangent lines of f at u, these slopes increasing as u increases from u_L to u_R. The shock waves, on the other hand, have a speed given by the slope of the chord between u_L and $u_R < U_L$ because

$$s = \frac{[f]}{[u]}.$$

This picture for getting the speeds of states that solve the Riemann problem from the graph of f applies to any scalar conservation law of form

$$u_t + f(u)_x = 0,$$

subject to $f''(u) > 0$, the condition for GN. It only remains to discuss the *admissibility* condition, or *entropy* condition that rules out the shock waves when $u_L < u_R$, allowable by the RH condition alone.

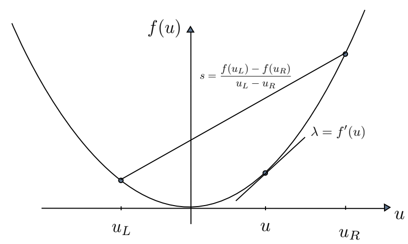

Figure 5: Riemann Problem for Scalar Conservation Law $u_t + f(u)_x = 0$

• The main condition that picks out the shocks $u_L > u_R$ for Burgers equation is the so called *Lax entropy condition.* This states that characteristics should impinge on the shock from both sides, c.f. Figure 6. Mathematically, this states that an admissible shock should satisfy

$$\lambda_R < s < \lambda_L. \tag{12}$$

Essentially, this rules out shocks in which the wave speed can increase from u_L to u_R, because a rarefaction wave could replace the shock wave as a solution of the Riemann problem in this case. The shock wave is not allowed when a smoother solution exists. A shock that can be replaced by a rarefaction wave is called a rarefaction shock. Such shocks are unstable because, under small perturbation of the initial data, the solution would find the simple wave from u_L to u_R, not the shock wave—and the rarefaction wave is not a small perturbation of the shock wave.

In contrast to rarefaction shock, Lax shocks are highly stable. Since characteristics impinge on the shock, a small purturbation would send the solution constant along characteristics, right back into the shock. Moreover, we showed that when you linearize the constant state in Burgers equation, the purturbations evolve with almost the same wave speed. Thus all small perturbations of u_L and u_R will get swept into the shock wave as the characteristics impinge on the shock. In particular, admissible Lax shocks destroy information as they propagate...information about the past is lost as characteristics impinge on the shock. See Figure 6 for how all information about the initial data except u_L and u_R can be lost by characteristics impinging on the shock. It follows that when shock waves are present, the past cannot be recovered from the present, and the solutions are no longer time-reversible. One can show that this loss of information in the compressible Euler equations corresponds to increase of entropy, a thermodynamical measure of information lost.

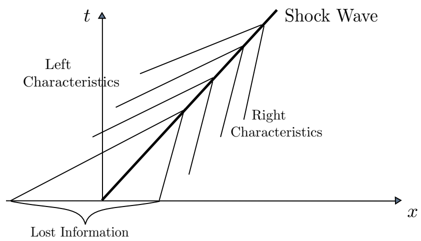

Figure 6: Characteristics Impinge on a Lax Shock With Consequent Loss of Information

4. The Riemann Problem for the p-system

We now solve the Riemann problem for the p-system under the assumptions that

$$p'(v) < 0, \quad p''(v) > 0. \tag{13}$$

The first condition just states that pressure rises with density $\rho = 1/v$, and the second guarantees genuine nonlinearity in both characteristic fields, c.f. Section 8. In particular, the isothermal equations of state

$$p = \frac{\sigma^2}{v}$$

meets both conditions (13).

To start, recall that the p-system is the nonlinear wave equation with sound speed $c(v) = \sqrt{-p'(v)}$, and is obtained from the compressible Euler equations by changing the spatial coordinate to a frame moving with the fluid at each point. The p-system takes the conservation form

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(v) \end{pmatrix}_x = 0., \tag{14}$$

of form (10) with conserved quantities $U = (v, u)$ and flux $f(U) = (-u, p(v))$. The p-system is physically equivalent to the compressible Euler equations, and in particular it can be shown that the RH jump conditions determine the same shock curves with transformed shock speeds. Because it is simpler, we now derive the shock curves for the p-system, and see how they connect with the rarefaction curves $\mathbf{R}^+$ derived in the previous section.

To construct the shock curves $\mathbf{S}(U_L)$, we solve for the set of right states $U = (v, u)$ that meet the RH jump conditions $s[U] = [f(U)]$ with left state $U_R = (v_L, u_R)$, and some speed s. This defines the so called *Hugoniot locus* of U_L, the set of all possible states that can be connected to U_L across a shock wave. Putting (14) into RH gives

$$s[U] - [f(U)] = s \begin{pmatrix} v - v_L \\ u - u_L \end{pmatrix} - \begin{pmatrix} u_L - u \\ p(v) - p(v_L) \end{pmatrix} = 0,$$

yielding the two equations

$$\begin{aligned} s(v - v_L) &= -(u - u_L), \\ s(u - u_L) &= p(v) - p(v_L). \end{aligned} \tag{15}$$

Multiplying both sides of (15) and (16) together gives

$$s^2 = -\frac{p(v) - p(v_L)}{v - v_L} = -p'(v_*), \tag{16}$$

where the mean value theorem gives the existence of $v_* = v_*(v_L, v)$ between v_L and v. (See Figure 7b.)

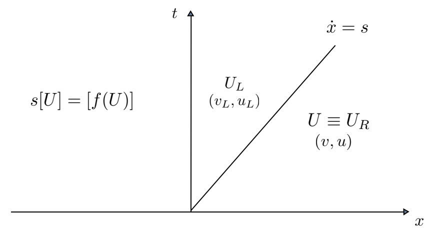

Figure 7a: States U in the Hugoniot Locus

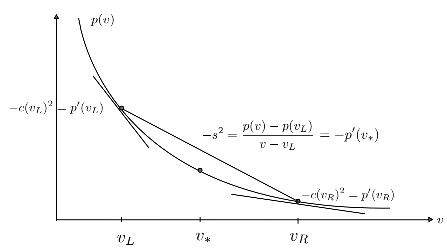

Figure 7b: Wave Speeds in Terms of the Pressure

Thus the shock speeds are

$$s = \pm\sqrt{-\frac{p(v)-p(v_L)}{v-v_L}} = \pm\sqrt{-p'(v_*)}, \tag{17}$$

Since

$$\lim_{v\to v_L} \sqrt{-p'(v_*)} = \pm\sqrt{-p'(v_L)} = \pm c(v_L), \tag{18}$$

we set the 1-shock speed equal to

$$s_1 \equiv s_1(v_L, v) = -\sqrt{-p'(v_*)}, \tag{19}$$

and the 2-shock speed equal to

$$s_2 \equiv s_2(v_L, v) = +\sqrt{-p'(v_*)}. \tag{20}$$

To obtain the shock curves, eliminate s from (15), (16) to obtain

$$(u-u_L)^2 = -(p(v)-p(v_L))(v-v_L), \tag{21}$$

or

$$\begin{aligned} u &= u_L \pm \sqrt{-(p(v)-p(v_L))(v-v_L)} \\ &= u_L \pm \sqrt{-\frac{p(v)-p(v_L)}{v-v_L}(v-v_L)} \\ &= u_L \pm \sqrt{-p'(v_*)}(v-v_L). \end{aligned}$$

Note that because $p'(v) < 0$, all functions under square root signs are positive.

Now define the 1-shock curve $\mathbf{S}_1$ by,

$$u = u_L + \sqrt{-(p(v)-p(v_L))(v-v_L)}$$

or

$$\mathbf{S_1}: \quad u = u_L + \sqrt{-p'(v_*)}(v-v_L).$$

and the 2-shock curve $\mathbf{S}_2$ as

$$u = u_L - \sqrt{-(p(v)-p(v_L))(v-v_L)},$$

or

$$\mathbf{S_2}: \quad u = u_L + \sqrt{-p'(v_*)}(v - v_L).$$

Now it is clear that $\mathbf{S}_1$ gives the curve of states U that make shocks with U_L of negative speed s_1, and $\mathbf{S}_2$ gives the curve of states U that make shocks with U_L of positive speed s_2. To see this, note that by (15),

$$s = -\frac{u - u_L}{v - v_L},$$

so when $s = s_1 < 0$ we have

$$-\frac{u - u_L}{v - v_L} < 0,$$

meaning we must be on $\mathbf{S}_1$, and when $s = s_2$, we have

$$-\frac{u - u_L}{v - v_L} > 0,$$

meaning we must be on $\mathbf{S}_2$.

Now recall

$$\mathbf{R}_1(U_L) = \mathbf{R}_1(U_L)^- \cup \mathbf{R}_1^+(U_L),$$

$$\mathbf{R}_2(U_L) = \mathbf{R}_2^-(U_L) \cup \mathbf{R}_2^+(U_L),$$

where $\mathbf{R}_1^+(U_L)$ and $\mathbf{R}_2^+(U_L)$ are the 1- and 2- rarefaction curves passing through the state U_L, that portion of the integral curves along which the eigenvalues increase, so the portion that corresponds to viable right states for rarefaction waves starting with left state U_L.

Similarly, define

$$\mathbf{S}_1(U_L) = \mathbf{S}_1(U_L)^- \cup \mathbf{S}_1^+(U_L),$$

$$\mathbf{S}_2(U_L) = \mathbf{S}_2^-(U_L) \cup \mathbf{S}_2^+(U_L),$$

as diagrammed in Figure 8. We now prove the following theorem:

Theorem 1. *For each U_L, the 1-shock curve $\mathbf{S}_1(U_L)$ intersects $\mathbf{R}_1(U_L)$ at the state U_L, at which point the two curves have equal first and second derivative. Similarly, the 2-shock curve $\mathbf{S}_2(U_L)$ intersects $\mathbf{R}_2(U_L)$ at the state U_L, and these latter two curves are also have equatl first and second derivatives at U_L. We say that the shock curves have C^2 tangency with the rarefaction curves at U_L.*

The tangency of the shock and rarefaction curves is diagrammed in Figure 9. The theorem completes the picture of the shock and rarefaction curves. I.e., the shock curve $\mathbf{S}_1^-(U_L)$ completes the rarefaction curve $\mathbf{R}_1^+(U_L)$ to a C^2 1-wave curve $\mathbf{W}_1(U_L)$ defined by, (c.f. Figure 10),

$$\mathbf{W}_1(U_L) = \mathbf{S}_1^-(U_L) \cup \mathbf{R}_1^+(U_L).$$

Similarly, define 2-wave curve $\mathbf{W}_1(U_L)$, also C^2, is defined by

$$\mathbf{W}_1(U_L) = \mathbf{S}_1^-(U_L) \cup \mathbf{R}_1^+(U_L).$$

Moreover, it follows from Figure 7b that the i-characteristics impinge on the i-shock waves like Figure 6, thereby meeting the Lax admissibility for shock waves, precisely when U_R is on $\mathbf{S}_i^-$. Finally, it is easy to see that the wave curves stay within the physical domain $v > 0$.

Before giving the proof of Theorem 1, we can complete the resolution of the Riemann problem by defining a coordinate system of wave curves based at U_L. For this, draw all 2-wave curves $W_2(U_M)$ for $U_M \in \mathbf{W}_1(U_L)$, as diagrammed in Figure 11. Then for any right state $U = U_R$, we can solve the Riemann problem by finding the state U_M such that $U_L \in \mathbf{W}_2(U_M)$. Then the Rieman problem is resolved by the negative speed 1-wave from U_L to U_M, followed by a positive speed 2-wave from U_M to U_L. The waves are either shock waves or rarefaction waves depending on which region (I-IV) U_R lies in relative to the coordinate system of wave curves at U_L, (see Figure 11). The various cases are diagrammed in Figures 12-15. Examples of wave interactions that can be resolved by Riemann problems alone are diagrammed in Figures 16,17. It is not so difficult to justify that for isothermal gas dynamics $p = \sigma^2/v$, this procedure produces a unique solution of the Riemann problem in the class of shock waves and rarefaction waves. A complete proof of uniqueness of solutions would entail showing that two wave curves $W_2(U_{M1})$ never intersects $W_2(U_{M2}$ for $U_{M1} \neq U_{M2}$ on $W_1(U_L)$. This is true for any p satisfying $p'(v) < 0$, $p''(v) > 0$. To prove existence of a solution for every U_R, entails proving that every state U_R in the physical domain $v > 0$, can be reached by $W_2(U_M)$, for some U_M on $W_1(U_L)$. It is not so difficult to show that this is true for the isothermal equation of state. But for more general equations of state, (like polytropic gases satisfying $p = 1/v^\gamma$, $\gamma > 1$), this is not strictly true because of the possible formation of vacuum states. See [Smoller] for a more in depth discussion of the very interesting issue of the vacuum.

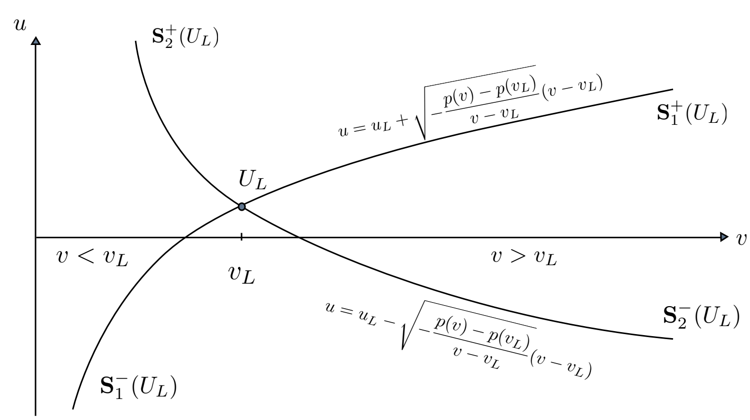

Figure 8: The Shock Curves for the p-system.

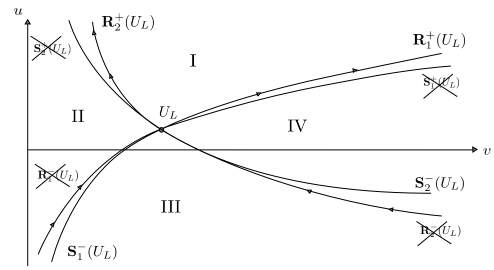

Figure 9: The Shock and Rarefaction Curves for the p-system.

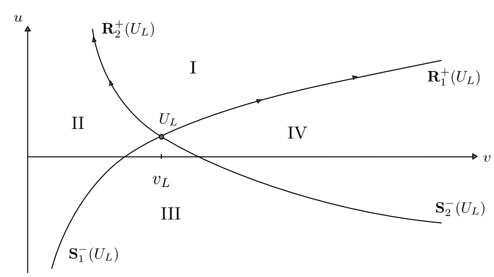

Figure 10: The Wave Curves of the p-system

• **Proof of Theorem 1:** We show that the 1-rarefaction curve is tangent to the 1-shock curve at $U = U_L$, the case of 2 shocks being similar. Recall then that the 1-integral curves are given by

$$s = u - h(v) = const,$$

so

$$u = h(v) + const, \tag{22}$$

where

$$h'(v) = c(v) = \sqrt{-p'(v)}.$$

The 1-shock curve is given by

$$u = u_L + \sqrt{-\frac{p(v) - p(v_L)}{v - v_L}}(v - v_L). \tag{23}$$

We check agreement of the first two derivatives at $v = v_L$. First, along (22),

$$\frac{du}{dv} = h'(v) = c(v), \qquad \frac{d^2u}{dv^2} = c'(v),$$

so at $v = v_L$ we have

$$\frac{du}{dv} = c(v_L), \qquad \frac{d^2u}{dv^2} = c'(v_L). \tag{24}$$

On the other hand, for the shock curve (23) we compute

$$\frac{du}{dv} = \frac{d}{dv}\sqrt{-\frac{p(v) - p(v_L)}{v - v_L}} \cdot (v - v_L) + \sqrt{-\frac{p(v) - p(v_L)}{v - v_L}}$$

so at $v = v_L$ we have

$$\frac{du}{dv} = \lim_{v \to v_L} \sqrt{-\frac{p(v) - p(v_L)}{v - v_L}} = \sqrt{-p'(v_L)} = c(v_L), \tag{25}$$

in agreement with (25). To verify agreement at the second derivative, differentiate (26) by the product rule to obtain

$$\frac{d^2u}{dv^2} = \frac{d^2}{dv^2}\sqrt{\cdot}(v - v_L) + 2\frac{d}{dv}\sqrt{\cdot}.$$

Since the first term will vanish when $v = v_L$, we get

$$\frac{d^2u}{dv^2}|_{v=v_L} = \lim_{v\to v_L} 2\frac{d}{dv}\sqrt{-\frac{p(v) - p(v_L)}{v - v_L}}.$$

But

$$2\frac{d}{dv}\sqrt{-\frac{p(v) - p(v_L)}{v - v_L}} = \frac{-1}{\sqrt{\cdot}}\frac{(v - v_L)p'(v) - \frac{(p(v)-p(v_L))}{v-v_L}}{v - v_L}.$$

Substituting the Taylor approximation

$$p(v)-p(v_L) = p'(v_L)(v-v_L)+\frac{1}{2}p''(v_l)(v-v_L)^2+O(v-v_L)^3,$$

and simplifying gives the second derivative of the shock curve at $v = v_L$ as

$$\frac{d^2u}{dv^2}|_{v=v_L} = \lim_{v\to v_L} -\frac{1}{\sqrt{\cdot}}\frac{1}{2}p''(v_L) = c'(v_L),$$

in agreement with (25), as claimed. $\square$

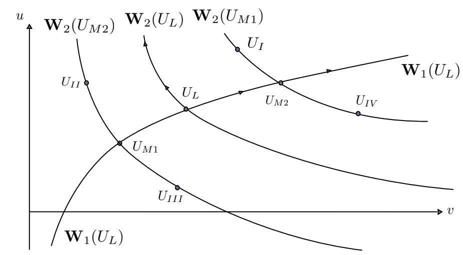

Figure 11: The Coordinate System of Wave Curves at U_L

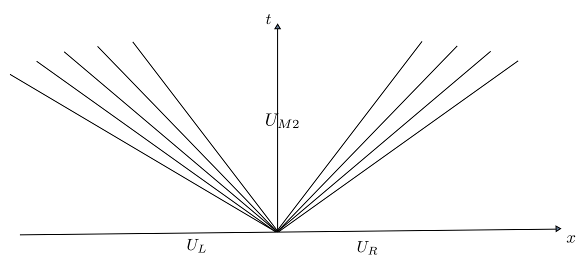

Figure 12: The Riemann Solution for Region I

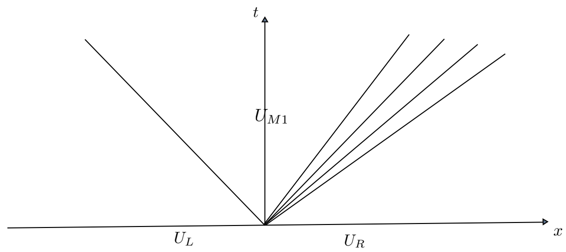

Figure 13: The Riemann Solution for Region II

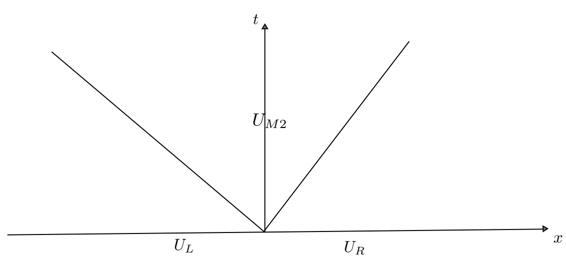

Figure 14: The Riemann Solution for Region III

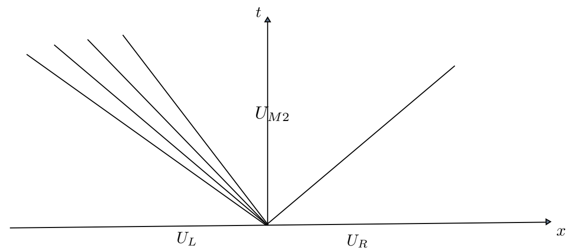

Figure 15: The Riemann Solution for Region IV

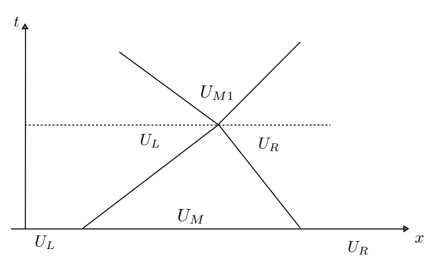

Figure 16: A 1-Shock Interacts With a 2-Shock

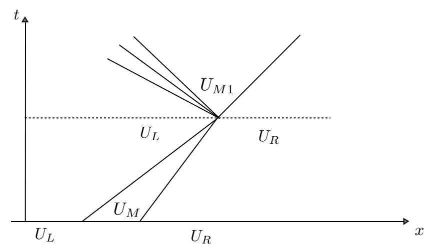

Figure 17: Two 2-Shocks Reflect a 1-rarefactions wave