Derivation of the Navier–Stokes equations

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(Redirected from Navier-Stokes equations/Derivation)

The intent of this article is to highlight the important points of the **derivation of the Navier–Stokes equations** as well as the application and formulation for different families of fluids.

Contents

Basic assumptions

The Navier–Stokes equations are based on the assumption that the fluid, at the scale of interest, is a continuum, in other words is not made up of discrete particles but rather a continuous substance. Another necessary assumption is that all the fields of interest like pressure, velocity, density, temperature and so on are differentiable, weakly at least.

The equations are derived from the basic principles of conservation of mass, momentum, and energy. For that matter, sometimes it is necessary to consider a finite arbitrary volume, called a control volume, over which these principles can be applied. This finite volume is denoted by Ω and its bounding surface $\partial\Omega$. The control volume can remain fixed in space or can move with the fluid.

The material derivative

Main article: material derivative

Changes in properties of a moving fluid can be measured in two different ways. One can measure a given property by either carrying out the measurement on a fixed point in space as particles of the fluid pass by, or by following a parcel of fluid along its streamline. The derivative of a field with respect to a fixed position in space is called the *Eulerian* derivative while the derivative following a moving parcel is called the *convective* or *material* derivative.

The material derivative is defined as the operator:

$$\frac{D}{Dt} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

where $\mathbf{v}$ is the velocity of the fluid. The first term on the right-hand side of the equation is the ordinary Eulerian derivative (i.e. the derivative on a fixed reference frame, representing changes at a point with respect to time) whereas the second term represents changes of a quantity with respect to position (see advection). This "special" derivative is in fact the ordinary derivative of a function of many variables along a path following the fluid motion; it may be derived through application of the chain rule.

For example, the measurement of changes in wind velocity in the atmosphere can be obtained with the help of an anemometer in a weather station or by mounting it on a weather balloon. The anemometer in the first case is measuring the velocity of all the moving particles passing through a fixed point in space, whereas in the second case the instrument is measuring changes in velocity as it moves with the fluid.

Conservation laws

The Navier–Stokes equation is a special case of the (general) continuity equation. It, and associated equations such as mass continuity, may be derived from conservation principles of:

- Mass
- Momentum
- Energy.

This is done via the Reynolds transport theorem, an integral relation stating that the sum of the changes of some extensive property (call it L) defined over a control volume Ω must be equal to what is lost (or gained) through the boundaries of the volume plus what is created/consumed by sources and sinks inside the control volume. This is expressed by the following integral equation:

$$\frac{d}{dt}\int_{\Omega} L \; dV = -\int_{\partial\Omega} L\mathbf{v}\cdot\mathbf{n} \; dA - \int_{\Omega} Q \; dV$$

where **v** is the velocity of the fluid and Q represents the sources and sinks in the fluid. Recall that Ω represents the control volume and $\partial\Omega$ its bounding surface.

The divergence theorem may be applied to the surface integral, changing it into a volume integral:

$$\frac{d}{dt}\int_{\Omega} L \; dV = -\int_{\Omega} \nabla\cdot(L\mathbf{v}) \; dV - \int_{\Omega} Q \; dV.$$

Applying Leibniz's rule to the integral on the left and then combining all of the integrals:

$$\int_\Omega \frac{\partial L}{\partial t}\, dV = -\int_\Omega \nabla\cdot(L\mathbf{v})\, dV - \int_\Omega Q\, dV \qquad \Rightarrow \qquad \int_\Omega \left(\frac{\partial L}{\partial t} + \nabla\cdot(L\mathbf{v}) + Q\right) dV = 0.$$

The integral must be zero for **any** control volume; this can only be true if the integrand itself is zero, so that:

$$\frac{\partial L}{\partial t} + \nabla\cdot(L\mathbf{v}) + Q = 0.$$

From this valuable relation (a very generic continuity equation), three important concepts may be concisely written: conservation of mass, conservation of momentum, and conservation of energy. Validity is retained if L is a vector, in which case the vector-vector product in the second term will be a dyad.

Conservation of momentum

The most elemental form of the Navier–Stokes equations is obtained when the conservation relation is applied to momentum. Writing momentum as $\rho\mathbf{v}$ gives:

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \mathbf{Q} = 0$$

where $\mathbf{v}\mathbf{v}$ is a dyad, a special case of tensor product, which results in a second rank tensor; the divergence of a second rank tensor is again a vector (a first rank tensor)[1]. Noting that a body force (notated $\mathbf{b}$) is a source or sink of momentum (per volume) and expanding the derivatives completely:

$$\mathbf{v}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\mathbf{v}\cdot\nabla\rho + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \rho\mathbf{v}\nabla\cdot\mathbf{v} = \mathbf{b}$$

Note that the gradient of a vector is a special case of the covariant derivative, the operation results in second rank tensors[1]; except in Cartesian coordinates, it's important to understand that this isn't simply an element by element gradient. Rearranging and recognizing that $\mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = \nabla\cdot(\rho\mathbf{v})$:

$$\mathbf{v}\left(\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v}\right) + \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{b}$$

$$\mathbf{v}\left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) + \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{b}$$

The leftmost expression enclosed in parentheses is, by mass continuity (shown in a moment), equal to zero. Noting that what remains on the left side of the equation is the convective derivative:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{b} \qquad \Rightarrow \qquad \rho\frac{D\mathbf{v}}{Dt} = \mathbf{b}$$

This appears to simply be an expression of Newton's second law (**F** = m**a**) in terms of body forces instead of point forces. Each term in any case of the Navier–Stokes equations is a body force. A shorter though less rigorous way to arrive at this result would be the application of the chain rule to acceleration:

$$\rho\frac{d}{dt}(\mathbf{v}(x,y,z,t)) = \mathbf{b} \qquad \Rightarrow \qquad \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt}\right) = \mathbf{b} \qquad \Rightarrow$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + u\frac{\partial \mathbf{v}}{\partial x} + v\frac{\partial \mathbf{v}}{\partial y} + w\frac{\partial \mathbf{v}}{\partial z}\right) = \mathbf{b} \qquad \Rightarrow \qquad \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{b}$$

where $\mathbf{v} = (u, v, w)$. The reason why this is "less rigorous" is that we haven't shown that picking $\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ is correct; however it does make sense since with that choice of path the derivative is "following" a fluid "particle", and in order for Newton's second law to work, forces must be summed following a particle. For this reason the convective derivative is also known as the particle derivative.

Conservation of mass

Mass may be considered also. Taking $Q = 0$ (no sources or sinks of mass) and putting in density:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

where ρ is the mass density (mass per unit volume), and $\mathbf{v}$ is the velocity of the fluid. This equation is called the **mass continuity equation**, or simply "the" continuity equation. This equation generally accompanies the Navier–Stokes equation.

In the case of an incompressible fluid, ρ is a constant and the equation reduces to:

$$\nabla \cdot \mathbf{v} = 0$$

which is in fact a statement of the conservation of volume.

General form of the equations of motion

The generic body force $\mathbf{b}$ seen previously is made specific first by breaking it up into two new terms, one to describe forces resulting from stresses and one for "other" forces such as gravity. By examining the forces acting on a small cube in a fluid, it may be shown that

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}$$

where $\boldsymbol{\sigma}$ is the stress tensor, and $\mathbf{f}$ accounts for other body forces present. This equation is called the Cauchy momentum equation and describes the non-relativistic momentum conservation of *any* continuum that conserves mass. $\boldsymbol{\sigma}$ is a rank two symmetric tensor given by its covariant components:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

where the σ are normal stresses and τ shear stresses. This tensor is split up into two terms:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \sigma_{xx}+p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}+p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}+p \end{pmatrix} = -pI + \mathbb{T}$$

where I is the 3 x 3 identity matrix and $\mathbb{T}$ is the deviatoric stress tensor. Note that the pressure *p* is equal to minus the mean normal stress:[2]

$$p = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right).$$

The motivation for doing this is that pressure is typically a variable of interest, and also this simplifies application to specific fluid families later on since the rightmost tensor $\mathbb{T}$ in the equation above must be zero for a fluid at rest. Note that $\mathbb{T}$ is traceless. The Navier–Stokes equation may now be written in the most general form:

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla\cdot\mathbb{T} + \mathbf{f}$$

This equation is still incomplete. For completion, one must make hypotheses on the form of $\mathbb{T}$, that is, one needs a constitutive law for the stress tensor which can be obtained for specific fluid families; additionally, if the flow is assumed compressible an equation of state will be required, which will likely further require a conservation of energy formulation.

Application to different fluids

The general form of the equations of motion is not "ready for use", the stress tensor is still unknown so that more information is needed; this information is normally some knowledge of the viscous behavior of the fluid. For different types of fluid flow this results in specific forms of the Navier–Stokes equations.

Newtonian fluid

Main article: Newtonian fluid

Compressible Newtonian fluid

The formulation for Newtonian fluids stems from an observation made by Newton that, for most fluids,

$$\tau \propto \frac{\partial u}{\partial y}$$

In order to apply this to the Navier–Stokes equations, three assumptions were made by Stokes:

- The stress tensor is a linear function of the strain rates.
- The fluid is isotropic.
- For a fluid at rest, $\nabla \cdot \mathbb{T}$ must be zero (so that hydrostatic pressure results).

Applying these assumptions will lead to:

$$\mathbb{T}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \nabla \cdot \mathbf{v}$$

δ_{ij} is the Kronecker delta. μ and λ are proportionality constants associated with the assumption that stress depends on strain linearly; μ is called the first coefficient of viscosity (usually just called "viscosity") and λ is the second coefficient of viscosity (related to bulk viscosity). The value of λ, which produces a viscous effect associated with volume change, is very difficult to determine, not even its sign is known with absolute certainty. Even in compressible flows, the term involving λ is often negligible; however it can occasionally be important even in nearly incompressible flows and is a matter of controversy. When taken nonzero, the most common approximation is $\lambda \approx -\frac{2}{3}\mu$.[3]

A straightforward substitution of $\mathbb{T}_{ij}$ into the momentum conservation equation will yield the **Navier–Stokes equations for a compressible Newtonian fluid**:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left(2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \mathbf{v} \right) + \frac{\partial}{\partial y} \left(\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{\partial}{\partial z} \left(\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right) + \rho g$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left(\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left(2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \mathbf{v} \right) + \frac{\partial}{\partial z} \left(\mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right) + \rho g_y$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left(\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right) + \frac{\partial}{\partial y} \left(\mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left(2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \mathbf{v} \right) +$$

or, more compactly in vector form,

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p+\nabla\cdot\left(\mu\cdot\left(\nabla\mathbf{v}+\left(\nabla\mathbf{v}\right)^T\right)\right)+\nabla(\lambda\nabla\cdot\mathbf{v})+\rho\mathbf{g}$$

where the transpose has been used. Gravity has been accounted for as "the" body force, ie $\mathbf{f}=\rho\mathbf{g}$. The associated mass continuity equation is:

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

In addition to this equation, an equation of state and an equation for the conservation of energy is needed. The equation of state to use depends on context (often the ideal gas law), the conservation of energy will read:

$$\rho\frac{Dh}{Dt}=\frac{Dp}{Dt}+\nabla\cdot(k\nabla T)+\Phi$$

Here, h is the enthalpy, T is the temperature, and Φ is a function representing the dissipation of energy due to viscous effects:

$$\Phi=\mu\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right)+\lambda(\nabla\cdot\mathbf{v})^2$$

With a good equation of state and good functions for the dependence of parameters (such as viscosity) on the variables, this system of equations seems to properly model the dynamics of all known gases and most liquids.

Incompressible Newtonian fluid

For the special (but very common) case of incompressible flow, the momentum equations simplify significantly. Taking into account the following assumptions:

- Viscosity μ will now be a constant
- The second viscosity effect $\lambda=0$
- The simplified mass continuity equation $\nabla\cdot\mathbf{v}=0$

then looking at the viscous terms of the x momentum equation for example we have:

$$\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}+\lambda\nabla\cdot\mathbf{v}\right)+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)$$

$$=2\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial^2 v}{\partial y\,\partial x}+\mu\frac{\partial^2 u}{\partial z^2}+\mu\frac{\partial^2 w}{\partial z\,\partial x}$$

$$=\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial^2 u}{\partial z^2}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 v}{\partial y\,\partial x}+\mu\frac{\partial^2 w}{\partial z\,\partial x}$$

$$=\mu\nabla^2 u+\mu\frac{\partial}{\partial x}\cancelto{0}{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}=\mu\nabla^2 u$$

Similarly for the y and z momentum directions we have $\mu\nabla^2 v$ and $\mu\nabla^2 w$. o

Non-Newtonian fluids

Main article: Non-Newtonian fluid

A non-Newtonian fluid is a fluid whose flow properties differ in any way from those of Newtonian fluids. Most commonly the viscosity of non-Newtonian fluids is not independent of shear rate or shear rate history. However, there are some non-Newtonian fluids with shear-independent viscosity, that nonetheless exhibit normal stress-differences or other non-Newtonian behaviour. Many salt solutions and molten polymers are non-Newtonian fluids, as are many commonly found substances such as ketchup, custard, toothpaste, starch suspensions, paint, blood, and shampoo. In a Newtonian fluid, the relation between the shear stress and the shear rate is linear, passing through the origin, the constant of proportionality being the coefficient of viscosity. In a non-Newtonian fluid, the relation between the shear stress and the shear rate is different, and can even be time-dependent. The study of the non-Newtonian fluids is usually called rheology. A few examples are given here.

Bingham fluid