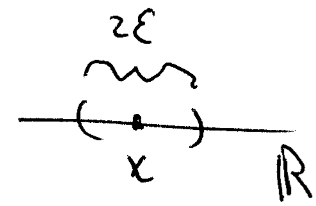
Compact Sets in Metric Space

Ex. Heine-Borel Thm: A subset of $\mathbb{R}^k$ is compact if and only if it is closed & bounded. ≈ Bolzano-Weierstrass in $\mathbb{R}^n$ generalizes to metric space S as Compact

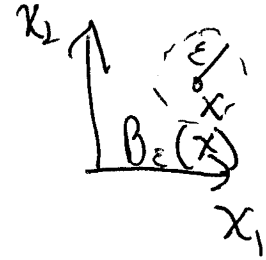
Topology of a metric space. (The set of open sets is called the topology)

- Defn: $B_\varepsilon(x)$ = "open ball of radius ε centered @ x"

$$B_\varepsilon(x) = \{ y \in S : d(x,y) < \varepsilon \}$$



- Let $E \subseteq S$. We say x is an interior point of E if $\exists\ \varepsilon > 0$ st $B_\varepsilon(x) \subseteq E$.



Defn: $E \subseteq S$ is open if every pt in E is an interior pt. (No boundary pts in E!)

Defn: E° = interior of E = set of all interior pts in E

<u>Defn</u>: E is <u>closed</u> if it is the complement of an open set, ie

$$E \text{ closed if } E^c \text{ is open}$$

$E^c = \{x \in S : x \notin E\}$ $\quad (E^c)^c = E$, $S^c = \emptyset$

<u>Thm</u>: The union of open sets is open & the intersection of closed sets is closed.

Pf. Assume E_λ $\lambda \in \Lambda$ are open. We show

$$\bigcup_\lambda E_\lambda \text{ is open.}$$

But $x \in E_\lambda \Rightarrow \exists\, B_\varepsilon(x) \subseteq E_\lambda \subseteq \bigcup_\lambda E_\lambda \Rightarrow$ every pt interior

De Morgan - $\quad \underbrace{\left(\bigcup_\lambda E_\lambda\right)^c}_{\text{open}} = \bigcap_\lambda \underbrace{E_\lambda^c}_{\text{open}}$

$\Rightarrow$ intersection of closed sets are closed.

<u>Thm</u>: The finite intersection ~~&~~ of open sets is open and finite ... of closed sets ~~are~~ is closed. (Same

<u>Thm</u>: E is closed iff the limit of every convergent sequence in E is also in E.

Pf. Assume E is closed & $x_n \to x_0$, $x_n \in E$. We prove $x_0 \in E$. If not then ~~the st~~ $x_0 \in E^c$ open $\Rightarrow \exists\, \varepsilon$ st $B_\varepsilon(x_0) \subseteq E^c$. But then $x_n \notin B_\varepsilon(x_0) \Rightarrow x_n \not\to x_0$ ✓

Assume the limit of every convergent sequence in E is also in E. We show E is closed. Assume E not closed. Then E^c is not open. Thus some element of E^c is <u>not</u> an interior pt $\Rightarrow \forall \varepsilon > 0\ \exists\, y \in E$ st $y \in B_\varepsilon(x)$.

Choose $\varepsilon = \frac{1}{n}$ & $y_n \in E$ st $y \in B_{1/n}(x)$. Then $y_n \to x$, $y_n \in E$, $x \notin E$ ※

Bolzano-Weierstrass Thm says — a bounded sequence in $\mathbb{R}^n$ admits a convergent subsequence.

Conclude: a bounded closed set E in $\mathbb{R}^n$ has the property that every sequence in E has a convergent subsequence that converges to a point in E.

Q: What property of a set $E \subseteq S$ in a general metric space guarantees that every sequence in E admits a convergent subsequence with limit in E?

Ans NOT closed & bounded (That's special to $\mathbb{R}^n$)

Ans Compact

<u>Defn</u>. We say a collection of open sets $\mathcal{O}_\lambda$ is a <u>covering</u> of E if

$$E \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda .$$

<u>Defn</u>. We say E is <u>compact</u> if every open covering of E <u>admits a finite subcover</u>.

<u>Theorem</u>: If $E \subseteq S$ is compact, then every sequence in E contains a convergent subsequence, with limit in E.

<u>Proof</u>. Assume E is <u>compact</u>. Let x_n be a sequence in E. We prove x_n admits a convergent subsequence $x_n \to x_0 \in E$. Assume $x \in E$ is not the limit of any subsequence from x_n. But then $\exists\, \varepsilon_x > 0$ s.t. $B_\varepsilon(x)$ contains x_n for only finite # of n. (OW, choose $\varepsilon = \frac{1}{k}$ & choose $x_{n_k} \in B_{1/k}(x) \Rightarrow x_{n_k} \to x$).

But $\bigcup_{x \in E} B_\varepsilon(x)$ is an open covering of E

$\Rightarrow \exists$ finite subcover $B_{\varepsilon_1}(x_1) \cup \cdots \cup B_{\varepsilon_N}(x_N) \supseteq E$.

But each $B_{\varepsilon_i}(x_i)$ contains x_n for only a finite # of n

$\Rightarrow x_n$ ~~only~~ lies in E for only a finite # of n ※

Defn: We say x_0 is a boundary point of $E \subseteq S$ if $\forall \varepsilon > 0$, $B_\varepsilon(x_0) \cap E \neq \emptyset$ & $B_\varepsilon(x_0) \cap E^c \neq \emptyset$. I.e., "there exist elements of E and E^c arbitrarily close to x_0."

Thm: $x \in \partial E$ iff $\exists$ sequences $y_n \in E$ & $z_n \in E^c$ st $y_n \to x$ & $z_n \to x$.

Defn: $\partial E \equiv$ set of all boundary pts of E.

Thm: $\partial E = E^- \setminus E^\circ \equiv E^- \cap (E^\circ)^c$

Pf. Assume $x \in \partial E$. Then $\forall \varepsilon > 0$, $B_\varepsilon(x)$ contains elements of $E^c \Rightarrow x \notin E^\circ$. But every closed set containing ~~E~~ must contain x because $\exists$ sequence $y_n \in E$ st $y_n \to x$, an all closed sets are closed under limits.

Assume $x \in \bar{E} \setminus E^\circ$. But $x \notin E^\circ \Rightarrow \forall \varepsilon > 0$ we must have $B_\varepsilon(x) \cap E^c \neq \emptyset$. It suffices only to prove $B_\varepsilon(x) \cap E \neq \emptyset \ \forall \varepsilon$ to conclude $x \in \partial E$. But if $x \in E$, then $B_\varepsilon(x) \cap E \neq \emptyset$, so assume $x \notin E$, and for some $\varepsilon > 0$, $B_\varepsilon(x) \cap E = \emptyset$. But then $E \subseteq B_\varepsilon(x)^c$ ~~open~~ closed & $x \notin B_\varepsilon(x)$ so x is not in the intersection of all closed sets containing E $\#$ $\therefore x \in \partial E$ ✓