Introduction

Brief Survey of Logic —

- A set is a collection of objects

  $\mathbb{N}$ = set of natural numbers

  $= \{1, 2, 3, \dots\}$ set given as a list

  $\mathbb{Z}$ = set of integers

  $= \{\dots -3, -2, -1, 0, 1, 2, 3 \dots\}$

  $\mathbb{R}$ = set of real #'s (to many to list)

  $\mathbb{Q}$ = set of rational #'s

  $$\mathbb{Q} = \left\{ \tfrac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\}$$

  such that

In general, a set is defined by:

$$X = \{ x \in S : P(x) \text{ is true} \}$$

read "X is the set of all x in S such that $P(x)$ is true"

Eg $\quad X = \{ p \in \mathbb{N} : \neg(\frac{p}{2} \in \mathbb{N}) \}$

" X equals the set of all p in $\mathbb{N}$ such that it is <u>not</u> the case that $\frac{p}{2} \in \mathbb{N}$ "

X = set of odd natural numbers.

- <u>Defn</u>: A <u>proposition</u> is a sentence that is either T or F
  statement

- <u>Compound Propositions</u> are formed using logical connectives —

$\neg P$ or $\sim P$ means not P

$P \wedge Q$ means P and Q

$P \vee Q$ means P or Q

$P \Longrightarrow Q$ means P implies Q $\Leftrightarrow$ if P then Q

P iff Q means $P \Longrightarrow Q$ and $Q \Longrightarrow P$.

The meaning of these statements is given by <u>truth tables</u>:

<u>Defn</u> A truth table for a statement involving P & Q tells the T or F of the statement as a function of the T or F of P & Q

<u>Eg</u>:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	P iff Q ($\Leftrightarrow$)
T	T	F	T	T	T	T
F	T	T	F	T	T	F
T	F	F	F	T	F	F
F	F	T	F	F	T	T

- Truth table can be taken as the <u>defn</u> of $\neg, \wedge, \vee, \rightarrow$, iff

- **Defn**: two propositions are <u>equivalent</u> if they have same truth table

Ex: Show $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are equivalent:

P.f.

P	Q	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	T	T
F	T	T	T
T	F	F	F
F	F	T	T

Defn: $\neg Q \Rightarrow \neg P$ is the <u>contrapositive</u> of $P \Rightarrow Q$

Defn: The <u>converse</u> of $P \Rightarrow Q$ is $Q \Rightarrow P$, different truth table.

Defn: tautology is always true $P \vee \neg P$

and contradiction is always false $\neg(P \wedge \neg P)$

<u>Defn</u>: $\Leftrightarrow$ is called the bi-conditional.

From truth table, it is clear that $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

& for truth asks that ~~means~~ P & Q are true or false together

◈ Axiomatic Method — <u>Assume</u> that a set of propositions are true. Find all true statements that follow from these by <u>logic</u>

A <u>Proof</u> of P from $A_1, \ldots, A_n$ is a <u>justification</u> that P must be true if we assume $A_1 \cdots A_n$ are true.

Said Differently: A proof of P is a demonstration that P follows from

The subject of mathematical logic attempts to systematize the rules of proof:

◇ <u>Examples</u> of valid proofs —

① modus ponens rule:

Given P and $P \Rightarrow Q$ are true, you may deduce Q is true

"If $\underbrace{x \in \mathbb{N} \text{ is even}}_{P(x)}$ then $\underbrace{\frac{x}{2} \in \mathbb{N}}_{Q(x)}$

If we know $P(x) \Rightarrow Q(x)$ is true $\forall x$ & we know $P(12)$ is true, then $Q(12)$ must be true.

② To prove $P \Rightarrow Q$

If you assump P true, and deduce Q using correct logic from known true statements, then you know $P \Rightarrow Q$ true

③ To prove $P \Rightarrow Q$ by contraposition

If you assume $\neg Q$ true & deduce $\neg P$, then you know $P \Rightarrow Q$ true

(ie. $\neg Q \Rightarrow \neg P$ is equiv to $P \Rightarrow Q$)

④ Proof by contradiction

If you assume $\neg P$ & deduce $\neg P \wedge P$, then P must be true (This is based on

5 To prove $P \Longleftrightarrow Q$ (P iff Q)

Prove $P \Rightarrow Q$ and prove $Q \Rightarrow P$.

- Quantifiers : $\forall$ ≡ for every

  $\exists$ ≡ there exist

- Defn: an open sentence or predicate is one that contains a variable

Eg: $P(x) \equiv$ "x is an element of $\mathbb{N}$"

$\equiv x \in \mathbb{N}$ (or "$x^2 = 2$")

The truth set of $P(x)$ is the set of all x that make $P(x)$ true.

The universe of discourse is the set that

Eg: The truth set of

$P(x)$ "$(x-1)(x+1)(x^2-2)=0$"

over $\mathbb{N}$ truth set $\{x \in \mathbb{N} : P(x)\} = \{1\}$

over $\mathbb{Z}$ truth set $\{x \in \mathbb{Z} : P(x)\} = \{1, -1\}$

over $\mathbb{R}$ truth set $\{x \in \mathbb{R} : P(x)\} = \{1, -1, \pm\sqrt{2}\}$

- **Defn**: two statements $P(x)$, $Q(x)$ are *equivalent* over U if their truth sets are equal over U. Two quantified statements are equivalent if their truth value is the same in *every universe*

- Universal/Existential statements —

  $(\exists x)P(x)$ is true if truth set of $P(x)$ is nonempty

  $(\forall x)P(x)$ is true if truth set of $P(x)$ is U

- Negating $\forall/\exists$ statements:

$$\neg(\exists x)P(x) \equiv \forall x(\neg P(x))$$

$$\neg(\forall x)P(x) \equiv \exists x(\neg P(x))$$

Proof Strategies for $\forall$ $\exists$ statements

1. To prove $(\forall x)\, P(x)$ directly

   Let $x_0 \in U$ be any fixed element of U.

   Prove $P(x_0)$ is true

2. To prove $(\exists x)\, P(x)$ directly

   Find $x_0 \in U$ st $P(x_0)$ is true

3. To Prove $(\forall x)\, P(x)$ by contradiction

   Assume $\neg(\forall x)\, P(x) \equiv \exists x\, (\neg P(x))$

   Assume $\neg P(x_0)$ is true for some x_0 and derive a contradiction

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4. ~~To prove~~ $\cancel{(\exists x)\, P(x)}$ ~~by contradiction~~

   ~~Assume~~

4 To prove $(\exists x)P(x)$ by contradiction:

Assume $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

and show $\neg P(x_0)$ leads to a $\#$

- Negating $\forall, \exists$ statements:

$$\neg(\exists x)P \equiv (\forall x)(\neg P)$$

$$\neg(\forall x)P \equiv (\exists x)(\neg P)$$

equiv statements

Ex: Prove the following statement by contradiction:

For every $\varepsilon > 0$, there exists an $N > 0$ such that $\varepsilon \cdot N > 1$.

Universe: $\varepsilon \in \mathbb{R}^+$, $N \in \mathbb{N}$

$$(\forall \varepsilon)(\exists N)\ P(\varepsilon, N)$$

$P(\varepsilon, N)$: " $\varepsilon \cdot N > 1$ "

So: we assume the negation, and derive a contradiction:

$$\neg(\forall \varepsilon)(\exists N)\, P(\varepsilon, N)$$

$$\equiv (\exists \varepsilon)\, \neg (\exists N)\, P(\varepsilon, N)$$

$$\equiv (\exists \varepsilon)\, \forall N\ \neg P(\varepsilon, N)$$

$\equiv$ there exists ε such that for evey N

$$\underbrace{\neg(\varepsilon N > 1)}_{\varepsilon N \leq 1}$$

Assuming this, let ε_0 be that ε. Then $\varepsilon_0 N \leq 1$ for all N means $N \leq \frac{1}{\varepsilon_0}$ for all N, but N is an unbounded set

- **Ex** $(\exists!\, x)P(x)$

"There exists a unique x st $P(x)$"
means the truth set of $P(x)$ consists of a single x.

Proof strategy:

① Find x_1 such that $P(x_1)$ true

② Assume x_2 is any other st $P(x_2)$ true

③ Prove: $x_1 = x_2$.