Write solutions on the paper provided. Put your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

Problem 1. (Short Answer: 32pts) Let \( s_n \) denote a sequence of real numbers. Give the precise mathematical definitions for the following:

(a) \( \lim_{n \to \infty} s_n = -2 \).

\[ \forall \epsilon > 0 \ \exists \ N \in \mathbb{N} \ st \ n > N \implies |s_n - (-2)| < \epsilon. \]

(b) \( \lim_{n \to \infty} s_n = -\infty \)

\[ \forall M > 0 \ \exists \ N \in \mathbb{N} \ st \ n > N \implies s_n < -M. \]

(c) Define what it means for \( s_n \) to be Cauchy.

\[ \forall \epsilon > 0 \ \exists \ N \in \mathbb{N} \ st \ n, m > N \implies |s_n - s_m| < \epsilon. \]

(d) Give the negation of the statement “\( s_n \) is Cauchy”.

\[ \exists \epsilon > 0 \ \forall \ N \in \mathbb{N} \ \exists n, m > N \ st \ |s_n - s_m| \geq \epsilon. \]

(e) Define \( s_N \) and \( \bar{s}_N \), the approximate lim inf and approximate lim sup of \( s_n \), and use these to define \( \underline{s} = \lim \inf s_n \) and \( \bar{s} = \lim \sup s_n \), respectively.

\[ s_N = \inf \{s_n : n > N\}; \ \bar{s}_N = \sup \{s_n : n > N\}. \ (\inf = \text{GLB}; \ \sup = \text{LUB}) \]

\[ \underline{s} = \lim_{N \to \infty} s_N; \ \bar{s} = \lim_{N \to \infty} \bar{s}_N \]

(f) Use \( \leq \) to give the correct inequalities that order the set \( \{s_N, \bar{s}_N, \underline{s}, \bar{s}\} \).

\[ s_N \leq \underline{s} \leq \bar{s} \leq \bar{s}_N \]
(g) Define what it means for a sequence to converge in a metric space $(S, d)$.

Definition: $s_n \to s_0$ in $(S, d)$ if

$$\forall \epsilon > 0 \ \exists \ N \in \mathbb{N} \ st \ n > N \ implies \ D(s_n, s) < \epsilon.$$ 

(h) Define the subsequential limit set $S$ of $s_n$, and identify $\text{Inf}\{S\}$ and $\text{Sup}\{S\}$.

Definition: The subsequential limit set of a set $S$ of a sequence $s_n$ is the set of all limits of convergent subsequences of $s_n$. In this case we have:

$$\text{Inf}\{S\} = \underline{s}, \ \text{Sup}\{S\} = \bar{s}.$$
Problem 2. (20pts) Use the fact that every natural number can be written uniquely as a product of prime factors to prove that $\sqrt{3}$ is not a positive rational number.

Proof by contradiction: Assume for contradiction that $a^3 = 3$ for some rational number $a = p/q$ in lowest terms. Assume $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_n$ are the prime factorizations of $p$ and $q$, containing no common factors by our assumption that $p/q$ is in lowest terms. Then $a^2 = 3$ says

$$\frac{p_1^2 \cdots p_n^2}{q_1^2 \cdots q_n^2} = 3$$

or

$$p_1^2 \cdots p_n^2 = 3q_1^2 \cdots q_n^2.$$ 

But since the LHS involves prime factors, it must be that 3 is among the primes $p_1, \ldots, p_n$. But that means $3^2$ is on the LHS, so 3 must be one of the factors $q_1, \ldots, q_m$ on the RHS. Then a factor of 3 cancels out in $p/q$, contradicting our assumption that $p/q$ is in lowest terms.
Problem 3. (20pts) Use the field axioms for the real numbers to prove that if \( a \in \mathbb{R} \), then \( a \cdot 0 = 0 \). (Give a field axiom reason for every step. Prove any lemma you use.)

Proof:

\[
\begin{align*}
0 &= (0 + 0), \text{ (defn additive inverse)} \\
a \cdot 0 &= a \cdot (0 + 0), \text{ (substitution)} \\
a \cdot 0 &= a \cdot 0 + a \cdot 0, \text{ (distributive property)} \\
a \cdot 0 + (-a \cdot 0) &= a \cdot 0 + a \cdot 0 + (-a \cdot 0), \text{ (exist of add inverse/add prop of equality)} \\
0 &= a \cdot 0, \text{ defn additive property of equality}
\end{align*}
\]
Problem 4. (24pts) Assume $s_n$ and $s_0$ are nonzero. Use the definition of convergence to give a direct proof that if $s_n \to s_0$, then $1/s_n \to 1/s_0$.

Proof: Assume $s_n \to s_0 \neq 0$, and $s_n \neq 0$. We prove $1/s_n \to 1/s_0$. Fix $\epsilon > 0$. We find $N \in \mathbb{N}$ such that, if $n > N$, then

$$|1/s_n - 1/s_0| = \frac{|s_0 - s_n|}{|s_n s_0|} < \epsilon.$$ 

Since $s_0 \neq 0$, and $s_n \to s_0$, we know that there exists $N_1$ such that $n > N_1$ implies $|s_n| > |s_0| - |s_0|/2 = |s_0|/2$. Now since $s_n \to s_0$, we can choose $N_2$ so that $n > N_2$ implies $|s_n - s_0| < \epsilon |s_0|^2/2$. Then letting $N = \text{Max}N_1, N_2$ implies

$$\frac{|s_0 - s_n|}{|s_n s_0|} < \frac{\epsilon |s_0|^2}{2} = \epsilon,$$

as claimed.
Problem 5. (20pts) Let $s_n = \sum_{k=1}^{n} a_k$ be the sequence of partial sums for infinite series $\sum_{k=1}^{\infty} a_k$, and let $t_n = \sum_{k=1}^{n} |a_k|.$

(a) Define what it means for the infinite series $s_n$ to converge. **Definition:** The series $\sum_{k=1}^{\infty} a_k$, converges if the sequence of partial sums $s_n$ converges.

(b) State the Cauchy criterion for convergence of the series $s_n$. **Cauchy Criterion Theorem:** The series $\sum_{k=1}^{\infty} a_k$, converges if and only if $\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ s.t \ n \geq m > N$ implies $|\sum_{k=m}^{n} a_k| < \epsilon.$

(c) Prove that if a series converges absolutely, then the series converges. **Absolute Convergence Implies Convergence:** Since $\sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} |a_k|,$ it follows that the Cauchys criterion for the absolute series gives $|\sum_{k=m}^{n} |a_k|| < \epsilon,$ and this implies $|\sum_{k=m}^{n} a_k| < \epsilon.$
Problem 6. (20pts) Prove that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

By the ratio test,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n + 1)!}{(n + 1)^{n+1}} \frac{n^n}{n!} = \frac{(n + 1)}{(n + 1)} \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n} = 1/e < 1.$$  

(One could also estimate

$$\frac{n!}{n^n} = \frac{n \cdot (n - 1) \cdot (n - 2) \cdots (1)}{n \cdot n \cdot n \cdots n} \leq \left(\frac{1}{2}\right)^{n/3}$$

by replacing $k$ by $n$ in the numerator for $k > n/2$, and by $n/2$ for $k < n$, say, to make it larger, so it can be compared to a geometric series for convergence.)
Problem 7. (20pts) Let $r$ be a real number such that $|r| < 1$, and let $s_n$ denote the sequence of partial sums

$$s_n = \sum_{k=m}^{n} r^k = r^m + r^{m+1} + r^{m+2} + \ldots + r^n.$$ 

(a) Derive a formula for $s_n$ that does not involve a summation, and use it to evaluate $\lim_{n \to \infty} s_n$.

Solution on the first Midterm!

(b) Prove that the repeating decimal $0.123123123\ldots$ is a rational number.

\[
0.123123123\ldots = \frac{123}{1000} + \frac{123}{1000^2} + \frac{123}{1000^3} + \ldots = 123 \sum_{k=1}^{\infty} \left( \frac{1}{1000} \right)^k = 123 \frac{1}{1000} \frac{1}{1 - \frac{1}{1000}},
\]

which is a rational number.
Problem 8. (24pts) Assume that \( x_n \to 0 \) and \( y_n \to 0 \) are convergent sequences of real numbers. Prove directly that \( \sqrt{x_n^2 + y_n^2} \to 0 \) converges.

We prove: \( \forall \epsilon > 0 \exists N \in \mathbb{N} \) st \( n > N \) implies \( \sqrt{x_n^2 + y_n^2} < \epsilon \). So fix \( \epsilon > 0 \). Choose \( N_1 \) so that for \( n > N_1 \) we have \( x_n^2 < \frac{\epsilon^2}{2} \). Choose \( N_2 \) for \( n > N_2 \) we have \( y_n^2 < \frac{\epsilon^2}{2} \). Set \( N = Max\{N_1, N_2\} \). Then \( n > N \) implies

\[
\sqrt{x_n^2 + y_n^2} < \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} = \epsilon,
\]

as claimed.
Problem 9. (20pts) Consider the sequence \( s_n = \{(−1)^n n + 1 + n\} \sin n \) of real numbers. Prove that \( s_n \) has a convergent subsequence. (You may use any theorem in the book.)

Proof: First note that if \( n \) is odd, then the sequence \( s_{2n} = \sin(2n) \). This subsequence of even terms is a bounded sequence of real numbers. Therefore the Bolzano-Weierstrass Theorem implies it has a convergent subsequence. Since a subsequence of a subsequence is also a subsequence of the original sequence, we have proven that the original sequence has a convergent subsequence.
Problem 10. (20pts) (Extra Credit) Let $a_n \geq 0$ be a sequence of positive real numbers, $n = 1, 2, 3, \ldots$, and let $p_n = \sum_{k=n}^{2n-1} a_k$. Assume that $p_n \to 0$. Does it follow that $\sum_{k=1}^{\infty} a_k$ converges? That is, does it follow that $\lim_{n \to \infty} \sum_{k=1}^{n} a_k$ converges to a real number? Prove your assertion, or else give a counterexample.

Counter-example: Consider the series $\sum_{k=1}^{\infty} a_k$ with $a_n = \frac{1}{n \ln n}$. This series diverges by the integral test: Namely, by the integral test, the series diverges or converges with the integral $\int_{e}^{\infty} \frac{dx}{x \ln(x)}$, and letting $u = \ln(x)$, $du = dx/x$, we have

$$\lim_{N \to \infty} \int_{e}^{N} \frac{dx}{x \ln(x)} = \lim_{N \to \infty} \int_{\ln(e)}^{\ln(N)} \frac{du}{u} = \ln(N) \to \infty.$$ 

However, $\sum_{k=n}^{2n-1} a_k \geq \int_{n}^{2n} \frac{dx}{x \ln(x)} \geq \frac{1}{\ln n} \int_{n}^{2n} \frac{dx}{x} = \frac{1}{\ln n} (\ln 2n - \ln n) = \frac{\ln 2n}{\ln n} \to 0$, verifying the counterexample.