FINAL EXAM
Math 25
Temple-F06

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Problem 1. (Short Answer: 32pts) Let s_n denote a sequence of real numbers. Give the precise mathematical definitions for the following:

(a) $\lim_{n\to\infty} s_n = -2$.

$$\forall \epsilon > 0 \ \exists \ N \in \mathcal{N} \ st \ n > N \ implies \ |s_n - (-2)| < \epsilon.$$

(b) $\lim_{n\to\infty} s_n = -\infty$

$$\forall M > 0 \ \exists \ N \in \mathcal{N} \ st \ n > N \ implies \ s_n < -M.$$

(c) Define what it means for s_n to be Cauchy.

$$\forall \epsilon > 0 \ \exists \ N \in \mathcal{N} \ st \ n, m > N \ implies \ |s_n - s_m| < \epsilon.$$

(d) Give the negation of the statement "s_n is Cauchy".

$$\exists \epsilon > 0 \ \forall \ N \in \mathcal{N} \ \exists \ n, m > N \ st \ |s_n - s_m| \geq \epsilon.$$

(e) Define $\underline{s}_N$ and $\bar{s}_N$, the approximate $\liminf$ and approximate $\limsup$ of s_n, and use these to define $\underline{s} = \liminf s_n$ and $\bar{s} = \limsup s_n$, respectively.

$$\underline{s}_N = inf\{s_n : n > N\}; \ \bar{s}_N = sup\{s_n : n > N\}. \ (inf = GLB, \ sup = LUB)$$

$$\underline{s} = lim_{N\to\infty} \ \underline{s}_N; \ \bar{s} = lim_{N\to\infty} \ \bar{s}_N$$

(f) Use $\leq$ to give the correct inequalities that order the set $\{\underline{s}_N, \bar{s}_N, \underline{s}, \bar{s}\}$.

$$\underline{s}_N \leq \underline{s} \leq \bar{s} \leq \bar{s}_N$$

(g) Define what it means for a sequence to converge in a *metric space* (S, d).

Definition: $s_n \to s_0$ in (S, d) if

$$\forall \epsilon > 0 \; \exists \; N \in \mathcal{N} \; st \; n > N \; implies \; D(s_n, s) < \epsilon.$$

(h) Define the subsequential limit set S of s_n, and identify $Inf\{S\}$ and $Sup\{S\}$.

Definition: The subsequential limit set of a set S of a sequence s_n is the set of all limits of convergent subsequences of s_n. In this case we have:

$$Inf\{S\} = \underline{s}, \;\; Sup\{S\} = \bar{s}.$$

Problem 2. (20pts) Use the fact that every natural number can be written uniquely as a product of prime factors to prove that $\sqrt{3}$ is not a positive rational number.

Proof by contradiction: Assume for contradiction that $a^3 = 3$ for some rational number $a = p/q$ in lowest terms. Assume $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_n$ are the prime factorizations of p and q, containing no common factors by our assumption that p/q is in lowest terms. Then $a^2 = 3$ says

$$\frac{p_1^2 \cdots p_n^2}{q_1^2 \cdots q_n^2} = 3$$

or

$$p_1^2 \cdots p_n^2 = 3q_1^2 \cdots q_n^2.$$

But since the LHS involves prime factors, it must be that 3 is among the primes $p_1, ..., p_n$. But that means 3^2 is on the LHS, so 3 must be one of the factors $q_1, ..., q_m$ on the RHS. Then a factor of 3 cancels out in p/q, contradicting our assumption that p/q is in lowest terms.

Problem 3. (20pts) Use the field axioms for the real numbers to prove that if $a \in \mathbf{R}$, then $a \cdot 0 = 0$. (Give a field axiom reason for every step. Prove any lemma you use.)

Proof:

$$\begin{aligned}
0 &= (0+0), \text{ (defn additive inverse)} \\
a \cdot 0 &= a \cdot (0+0), \text{ (substitution)} \\
a \cdot 0 &= a \cdot 0 + a \cdot 0, \text{ (distributive property)} \\
a \cdot 0 + (-a \cdot 0) &= a \cdot 0 + a \cdot 0 + (-a \cdot 0), \text{ (exist of add inverse/add prop of equality)} \\
0 &= a \cdot 0, \text{ defn additive property of equality)}
\end{aligned}$$

Problem 4. (24pts) Assume s_n and s_0 are nonzero. Use the definition of convergence to give a direct proof that if $s_n \to s_0$, then $1/s_n \to 1/s_0$.

Proof: Assume $s_n \to s_0 \neq 0$, and $s_n \neq 0$. We prove $1/s_n \to 1/s_0$. Fix $\epsilon > 0$. We find $N \in \mathcal{N}$ such that, if $n > N$, then

$$|1/s_n - 1/s_0| = \frac{|s_0 - s_n|}{|s_n s_0|} < \epsilon.$$

Since $s_0 \neq 0$, and $s_n \to s_0$, we know that there exists N_1 such that $n > N_1$ implies $|s_n| > |s_0| - |s_0|/2 = |s_0|/2$. Now since $s_n \to s_0$, we can choose N_2 so that $n > N_2$ implies $|s_n - s_0| < \epsilon \frac{|s_0|^2}{2}$. Then letting $N = Max N_1, N_2$ implies

$$\frac{|s_0 - s_n|}{|s_n s_0|} < \frac{\epsilon \frac{|s_0|^2}{2}}{\frac{|s_0|^2}{2} s} = \epsilon,$$

as claimed.

Problem 5. (20pts) Let $s_n = \sum_{k=1}^{n} a_k$ be the sequence of partial sums for infinite series $\sum_{k=1}^{\infty} a_k$, and let $t_n = \sum_{k=1}^{n} |a_k|$.

(a) Define what it means for the infinite series s_n to converge.

Definition: The series $\sum_{k=1}^{\infty} a_k$, converges if the sequence of partial sums s_n converges.

(b) State the Cauchy criterion for convergence of the series s_n.

Cauchy Criterion Theorem: The series $\sum_{k=1}^{\infty} a_k$, converges if and only if $\forall \epsilon > 0\ \exists N \in \mathcal{N} st\ n \geq m > N\ implies\ |\sum_{k=m}^{n} a_k| < \epsilon$.

(c) Prove that if a series converges absolutely, then the series converges.

Absolute Convergence Implies Convergence: Since $\sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} |a_k|$, it follows that the Cauchys criterion for the absolute series gives $|\sum_{k=m}^{n} |a_k|| < \epsilon$, and this implies $|\sum_{k=m}^{n} a_k| < \epsilon$

Problem 6. (20pts) Prove that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.
By the ratio test,

$$\frac{|a_{n+1|}}{|a_n|} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)}{(n+1)} \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} = 1/e < 1.$$

(One could also estimate

$$\frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (1)}{n \cdot n \cdot n \cdots n} \leq (1/2)^{n/3}$$

by replacing k by n in the numerator for $k > n/2$, and by $n/2$ for $k < n$, say, to make it larger, so it can be compared to a geometric series for convergence.)

Problem 7. (20pts) Let r be a real number such that $|r| < 1$, and let s_n denote the sequence of partial sums

$$s_n = \sum_{k=m}^{n} r^k = r^m + r^{m+1} + r^{m+2} + r^{m+3} + \cdots + r^n.$$

(a) Derive a formula for s_n that does not involve a summation, and use it to evaluate $\lim_{n\to\infty} s_n$.

Solution on the first Midterm!

(b) Prove that the repeating decimal .123123123... is a rational number.

$$.123123123... = \frac{123}{1000} + \frac{123}{1000^2} + \frac{123}{1000^3} + \cdots = 123 \sum_{k=1}^{\infty} \left(\frac{1}{1000}\right)^k = \frac{123}{1000} \frac{1}{1 - \frac{1}{1000}},$$

which is a rational number.

Problem 8. **(24pts)** Assume that $x_n \to 0$ and $y_n \to 0$ are convergent sequences of real numbers. Prove directly that $\sqrt{x_n^2 + y_n^2} \to 0$ converges.

We prove: $\forall \epsilon > 0 \exists\ N \in \mathcal{N}\ st\ n > N\ implies\ \sqrt{x_n^2 + y_n^2} < \epsilon$. So fix $\epsilon > 0$. Choose N_1 so that for $n > N_1$ we have $x_n^2 < \epsilon^2/2$. Choose N_2 for $n > N_2$ we have $y_n^2 < \epsilon^2/2$. Set $N = Max\{N_1, N_2\}$. Then $n > N$ implies

$$\sqrt{x_n^2 + y_n^2} < \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} = \epsilon,$$

as claimed.

Problem 9. (20pts) Consider the sequence $s_n = \{(-1)^n n + 1 + n\} \sin n$ of real numbers. Prove that s_n has a convergent subsequence. (You may use any theorem in the book.)

Proof: First note that if n is odd, then the sequence $s_{2n} = sin(2n)$. This subsequence of even terms is a bounded sequence of real numbers. Therefore the Bolzano-Weierstrass Theorem implies it has a convergent subsequence. Since a subsequence of a subsequence is also a subsequence of the original sequence, we have proven that the original sequence has a convergent subsequence.

Problem 10. (20pts) (Extra Credit) Let $a_n \geq 0$ be a sequence of positive real numbers, $n = 1, 2, 3...$, and let $p_n = \sum_{k=n}^{2n-1} a_k$. Assume that $p_n \to 0$. Does it follow that $\sum_{k=1}^{\infty} a_k$ converges? That is, does it follow that $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ converges to a real number? Prove your assertion, or else give a counterexample.

Counter-example: Consider the series $\sum_{k=1}^{\infty} a_n$ with $a_n = \frac{1}{n \ln n}$. This series diverges by the integral test: Namely, by the integral test, the series diverges or converges with the integral $\int_e^{\infty} \frac{dx}{x \ln(x)}$, and letting $u = \ln(x)$, $du = dx/x$, we have

$$\lim_{N\to\infty} \int_e^N \frac{dx}{x \ln(x)} = \lim_{N\to\infty} \int_{\ln e}^N \frac{du}{u} = \ln(N) \to \infty.$$

However, $\sum_{k=n}^{2n-1} a_k \geq \int_n^{2n} \frac{dx}{x \ln(x)} \geq \frac{1}{\ln n} \int_n^{2n} \frac{dx}{x} = \frac{1}{\ln n} (\ln 2n - \ln n) = \frac{\ln 2}{\ln n} \to 0$, verifying the counterexample.