Now we apply the induction hypothesis P_n to obtain

$$|\sin(n+1)x| \leq n|\sin x| + |\sin x| = (n+1)|\sin x|.$$

Thus P_{n+1} holds. Finally, the result holds for all n by mathematical induction. □

Exercises

1.1. Prove $1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all natural numbers n.

1.2. Prove $3 + 11 + \cdots + (8n-5) = 4n^2 - n$ for all natural numbers n.

1.3. Prove $1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2$ for all natural numbers n.

1.4. **(a)** Guess a formula for $1 + 3 + \cdots + (2n-1)$ by evaluating the sum for $n = 1, 2, 3$, and 4. [For $n = 1$, the sum is simply 1.]

(b) Prove your formula using mathematical induction.

1.5. Prove $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for all natural numbers n.

1.6. Prove that $(11)^n - 4^n$ is divisible by 7 when n is a natural number.

1.7. Prove that $7^n - 6n - 1$ is divisible by 36 for all positive integers n.

1.8. The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geq m$.

(a) Prove that $n^2 > n + 1$ for all integers $n \geq 2$.

(b) Prove that $n! > n^2$ for all integers $n \geq 4$. [Recall that $n! = n(n-1)\cdots 2 \cdot 1$; for example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.]

1.9. **(a)** Decide for which integers the inequality $2^n > n^2$ is true.

(b) Prove your claim in (a) by mathematical induction.

1.10. Prove $(2n+1) + (2n+3) + (2n+5) + \cdots + (4n-1) = 3n^2$ for all positive integers n.

1.11. For each $n \in \mathbf{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer."

(a) Prove that P_{n+1} is true whenever P_n is true.

(b) For which n is P_n actually true? What is the moral of this exercise?

1.12. For $n \in \mathbf{N}$, let $n!$ [read "n factorial"] denote the product $1 \cdot 2 \cdot 3 \cdots n$. Also let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for} \quad k = 0, 1, \ldots, n.$$

The *binomial theorem* asserts that

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots \\ &\quad + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= a^n + na^{n-1}b + \tfrac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.\end{aligned}$$

(a) Verify the binomial theorem for $n = 1$, 2, and 3.

(b) Show that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.

(c) Prove the binomial theorem using mathematical induction and part (b).

§2 The Set $\mathbb{Q}$ of Rational Numbers

Small children first learn to add and to multiply natural numbers. After subtraction is introduced, the need to expand the number system to include 0 and negative numbers becomes apparent. At this point the world of numbers is enlarged to include the set $\mathbb{Z}$ of all *integers*. Thus we have $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$.

Soon the space $\mathbb{Z}$ also becomes inadequate when division is introduced. The solution is to enlarge the world of numbers to include all fractions. Accordingly, we study the space $\mathbb{Q}$ of all *rational numbers*, i.e., numbers of the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. Note that $\mathbb{Q}$ contains all terminating decimals such as $1.492 = \frac{1492}{1000}$. The connection between decimals and real numbers is discussed in 10.3 and §16. The space $\mathbb{Q}$ is a highly satisfactory algebraic system in which the basic operations addition, multiplication, subtraction and division can be fully studied. No system is perfect, however, and $\mathbb{Q}$

and these all can be eliminated by mentally substituting them into the equation. We conclude that $4-7b^2$ cannot be rational, so b cannot be rational. ■

As a practical matter, many or all of the rational candidates given by the Rational Zeros Theorem can be eliminated by approximating the quantity in question [perhaps with the aid of a calculator]. It is nearly obvious that the values in Examples 2 through 5 are not integers, while all the rational candidates are. My calculator says that b in Example 6 is approximately .2767; the nearest rational candidate is $+2/7$ which is approximately .2857.

Exercises

2.1. Show that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

2.2. Show that $2^{1/3}$, $5^{1/7}$, and $(13)^{1/4}$ do not represent rational numbers.

2.3. Show that $(2+\sqrt{2})^{1/2}$ does not represent a rational number.

2.4. Show that $(5-\sqrt{3})^{1/3}$ does not represent a rational number.

2.5. Show that $[3+\sqrt{2}]^{2/3}$ does not represent a rational number.

2.6. In connection with Example 6, discuss why $4-7b^2$ must be rational if b is rational.

§3 The Set $\mathbb{R}$ of Real Numbers

The set $\mathbb{Q}$ is probably the largest system of numbers with which you really feel comfortable. There are some subtleties but you have learned to cope with them. For example, $\mathbb{Q}$ is not simply the *set* $\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$, since we regard some pairs of different looking fractions as equal. For example, $\frac{2}{4}$ and $\frac{3}{6}$ are regarded as the same element of $\mathbb{Q}$. A rigorous development of $\mathbb{Q}$ based on $\mathbb{Z}$, which in turn is based on $\mathbb{N}$, would require us to introduce the notion of equivalence class; see [38]. In this book we assume a familiarity with and understanding of $\mathbb{Q}$ as an algebraic system. However, in order

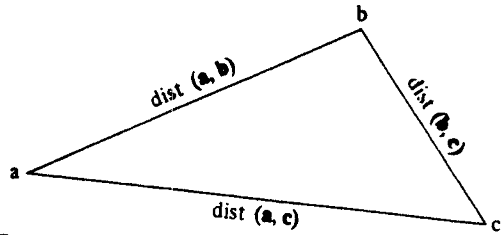

FIGURE 3.2

The inequality in Corollary 3.6 is very closely related to an inequality concerning points **a**, **b**, **c** in the plane, and the latter inequality can be interpreted as a statement about triangles: the length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides. See Figure 3.2. For this reason, the inequality in Corollary 3.6 and its close relative (iii) in 3.5 are often called the *Triangle Inequality*.

3.7 Triangle Inequality.
$|a+b| \le |a| + |b|$ for all a, b.

A useful variant of the triangle inequality is given in Exercise 3.5(b).

Exercises

3.1. **(a)** Which of the properties A1-A4, M1-M4, DL, O1-O5 fail for $\mathbb{N}$?

(b) Which of these properties fail for $\mathbb{Z}$?

3.2. **(a)** The commutative law A2 was used in the proof of (ii) in Theorem 3.1. Where?

(b) The commutative law A2 was also used in the proof of (iii) in Theorem 3.1. Where?

3.3. Prove (iv) and (v) of Theorem 3.1.

3.4. Prove (v) and (vii) of Theorem 3.2.

3.5. **(a)** Show that $|b| \le a$ if and only if $-a \le b \le a$.

(b) Prove that $||a| - |b|| \le |a - b|$ for all $a, b \in \mathbb{R}$.

3.6. **(a)** Prove that $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$. *Hint*: Apply the triangle inequality twice. Do *not* consider eight cases.

(b) Use induction to prove

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for n numbers $a_1, a_2, \ldots, a_n$.

3.7. **(a)** Show that $|b| < a$ if and only if $-a < b < a$.

(b) Show that $|a - b| < c$ if and only if $b - c < a < b + c$.

(c) Show that $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

3.8. Let $a, b \in \mathbb{R}$. Show that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

§4 The Completeness Axiom

In this section we give the completeness axiom for $\mathbb{R}$. This is the axiom that will assure us that $\mathbb{R}$ has no "gaps." It has far-reaching consequences and almost every significant result in this book relies on it. Most theorems in this book would be false if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers.

4.1 Definition.
Let S be a nonempty subset of $\mathbb{R}$.

(a) If S contains a largest element s_0 [that is, s_0 belongs to S and $s \leq s_0$ for all $s \in S$], then we call s_0 the *maximum of* S and write $s_0 = \max S$.

(b) If S contains a smallest element, then we call the smallest element the *minimum of* S and write it as $\min S$.

Example 1

(a) Every finite nonempty subset of $\mathbb{R}$ has a maximum and a minimum. Thus

$$\max\{1, 2, 3, 4, 5\} = 5 \quad \text{and} \quad \min\{1, 2, 3, 4, 5\} = 1,$$
$$\max\{0, \pi, -7, e, 3, 4/3\} = \pi \quad \text{and} \quad \min\{0, \pi, -7, e, 3, 4/3\} = -7,$$
$$\max\{n \in \mathbb{Z} : -4 < n \leq 100\} = 100 \quad \text{and}$$
$$\min\{n \in \mathbb{Z} : -4 < n \leq 100\} = -3.$$

Then $an < m$ but $m - 1 \le an$. Also, we have

$$m = (m-1) + 1 \le an + 1 < an + (bn - an) = bn,$$

so (1) holds. ■

Exercises

4.1. For each set below that is bounded above, list three upper bounds for the set. Otherwise write "NOT BOUNDED ABOVE" or "NBA."

- **(a)** $[0,1]$
- **(b)** $(0,1)$
- **(c)** $\{2,7\}$
- **(d)** $\{\pi, e\}$
- **(e)** $\{\frac{1}{n} : n \in \mathbb{N}\}$
- **(f)** $\{0\}$
- **(g)** $[0,1] \cup [2,3]$
- **(h)** $\cup_{n=1}^{\infty}[2n, 2n+1]$
- **(i)** $\cap_{n=1}^{\infty}[-\frac{1}{n}, 1+\frac{1}{n}]$
- **(j)** $\{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$
- **(k)** $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$
- **(l)** $\{r \in \mathbb{Q} : r < 2\}$
- **(m)** $\{r \in \mathbb{Q} : r^2 < 4\}$
- **(n)** $\{r \in \mathbb{Q} : r^2 < 2\}$
- **(o)** $\{x \in \mathbb{R} : x < 0\}$
- **(p)** $\{1, \frac{\pi}{3}, \pi^2, 10\}$
- **(q)** $\{0, 1, 2, 4, 8, 16\}$
- **(r)** $\cap_{n=1}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n})$
- **(s)** $\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\}$
- **(t)** $\{x \in \mathbb{R} : x^3 < 8\}$
- **(u)** $\{x^2 : x \in \mathbb{R}\}$
- **(v)** $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$
- **(w)** $\{\sin(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

4.2. Repeat Exercise 4.1 for lower bounds.

4.3. For each set in Exercise 4.1, give its supremum if it has one. Otherwise write "NO sup."

4.4. Repeat Exercise 4.3 for infima [plural of infimum].

4.5. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that if $\sup S$ belongs to S, then $\sup S = \max S$. *Hint*: Your proof should be very short.

4.6. Let S be a nonempty bounded subset of $\mathbb{R}$.

(a) Prove that $\inf S \le \sup S$. *Hint*: This is almost obvious; your proof should be short.

(b) What can you say about S if $\inf S = \sup S$?

4.7. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

(a) Prove that if $S \subseteq T$, then $\inf T \le \inf S \le \sup S \le \sup T$.

(b) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$. *Note*: In part (b), do *not* assume $S \subseteq T$.

4.8. Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \le t$ for all $s \in S$ and $t \in T$.

(a) Observe that S is bounded above and that T is bounded below.

(b) Prove that $\sup S \le \inf T$.

(c) Give an example of such sets S and T where $S \cap T$ is nonempty.

(d) Give an example of sets S and T where $\sup S = \inf T$ and $S \cap T$ is the empty set.

4.9. Complete the proof that $\inf S = -\sup(-S)$ in Corollary 4.5 by proving (1) and (2).

4.10. Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

4.11. Consider $a, b \in \mathbb{R}$ where $a < b$. Use Denseness of $\mathbb{Q}$ 4.7 to show that there are infinitely many rationals between a and b.

4.12. Let $\mathbb{I}$ be the set of real numbers that are not rational; elements of $\mathbb{I}$ are called *irrational numbers*. Prove that if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$. *Hint*: First show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$.

4.13. Prove that the following are equivalent for real numbers a, b, c. [*Equivalent* means that either all the properties hold or none of the properties hold.]

(a) $|a - b| < c$,

(b) $b - c < a < b + c$,

(c) $a \in (b - c, b + c)$.

Hint: Use Exercise 3.7(b).

4.14. Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let S be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a) Prove that $\sup S = \sup A + \sup B$.

(b) Prove that $\inf S = \inf A + \inf B$.

4.15. Let $a, b \in \mathbb{R}$. Show that if $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \le b$. Compare Exercise 3.8.

4.16. Show that $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

§5 The Sy

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Exercises

5.1. Write the following sets in interval notation:

(a) $\{x \in \mathbb{R} : x < 0\}$ **(b)** $\{x \in \mathbb{R} : x^3 \le 8\}$

(c) $\{x^2 : x \in \mathbb{R}\}$ **(d)** $\{x \in \mathbb{R} : x^2 < 8\}$

5.2. Give the infimum and supremum of each set listed in Exercise 5.1.

5.3. Give the infimum and supremum of each unbounded set listed in Exercise 4.1.

5.4. Let S be a nonempty subset of $\mathbb{R}$, and let $-S = \{-s : s \in S\}$. Prove that $\inf S = -\sup(-S)$. *Hint*: For the case $-\infty < \inf S$, simply state that this was proved in Exercise 4.9.

5.5. Prove that $\inf S \le \sup S$ for every nonempty subset of $\mathbb{R}$. Compare Exercise 4.6(a).

5.6. Let S and T be nonempty subsets of $\mathbb{R}$ such that $S \subseteq T$. Prove that $\inf T \le \inf S \le \sup S \le \sup T$. Compare Exercise 4.7(a).

§6 * A Development of $\mathbb{R}$

There are several ways to give a careful development of $\mathbb{R}$ based on $\mathbb{Q}$. We will briefly discuss one of them and give suggestions for further reading on this topic. [See the remarks about optional sections in the preface.]

To motivate our development we begin by observing that

$$a = \sup\{r \in \mathbb{Q} : r < a\} \quad \text{for each} \quad a \in \mathbb{R};$$

see Exercise 4.16. Note the intimate relationship: $a \le b$ if and only if $\{r \in \mathbb{Q} : r < a\} \subseteq \{r \in \mathbb{Q} : r < b\}$ and, moreover, $a = b$ if and only if $\{r \in \mathbb{Q} : r < a\} = \{r \in \mathbb{Q} : r < b\}$. Subsets α of $\mathbb{Q}$ having the form $\{r \in \mathbb{Q} : r < a\}$ satisfy these properties:

(i) $\alpha \ne \mathbb{Q}$ and α is not empty,
(ii) if $r \in \alpha$, $s \in \mathbb{Q}$ and $s < r$, then $s \in \alpha$,
(iii) α contains no largest rational.

Moreover, every subset α of $\mathbb{Q}$ that satisfies (i)–(iii) has the form $\{r \in \mathbb{Q} : r < a\}$ for some $a \in \mathbb{R}$; in fact, $a = \sup \alpha$. Subsets α of $\mathbb{Q}$ satisfying (i)–(iii) are called *Dedekind cuts*.

The remarks in the last paragraph relating real numbers and Dedekind cuts are based on our knowledge of $\mathbb{R}$, including the completeness axiom. But they can also motivate a development of $\mathbb{R}$ based solely on $\mathbb{Q}$. In such a development we make no *a priori* assumptions about $\mathbb{R}$. We assume only that we have the ordered field $\mathbb{Q}$ and that $\mathbb{Q}$ satisfies the Archimedean property 4.6. A Dedekind cut is a subset α of $\mathbb{Q}$ satisfying (i)–(iii). The set $\mathbb{R}$ of real numbers is *defined* as the space of all Dedekind cuts. Thus elements of $\mathbb{R}$ are *defined* as certain subsets of $\mathbb{Q}$. The rational numbers are identified with certain Dedekind cuts in the natural way: each rational s corresponds to the Dedekind cut $s^* = \{r \in \mathbb{Q} : r < s\}$. In this way $\mathbb{Q}$ is regarded as a subset of $\mathbb{R}$, that is, $\mathbb{Q}$ is identified with the set $\mathbb{Q}^* = \{s^* : s \in \mathbb{Q}\}$.

The set $\mathbb{R}$ defined in the last paragraph is given an order structure as follows: if α and β are Dedekind cuts, then we define $\alpha \leq \beta$ to signify that $\alpha \subseteq \beta$. Properties O1, O2 and O3 in §3 hold for this ordering. Addition is defined in $\mathbb{R}$ as follows: if α and β are Dedekind cuts, then

$$\alpha + \beta = \{r_1 + r_2 : r_1 \in \alpha \text{ and } r_2 \in \beta\}.$$

It turns out that $\alpha + \beta$ is a Dedekind cut [hence in $\mathbb{R}$] and that this definition of addition satisfies properties A1-A4 in §3. Multiplication of Dedekind cuts is a tedious business and has to be defined first for Dedekind cuts that are $\geq 0^*$. For a naive attempt, see Exercise 6.4. After the product of Dedekind cuts has been defined, the remaining properties of an ordered field can be verified for $\mathbb{R}$. The ordered field $\mathbb{R}$ constructed in this manner from $\mathbb{Q}$ is complete: the completeness property in 4.4 can be *proved* rather than taken as an axiom.

The development of $\mathbb{R}$ outlined above is given in [34] and [36]. The real numbers are developed from Cauchy sequences in $\mathbb{Q}$ in [23], §5. A thorough development of $\mathbb{R}$ based on Peano's axioms is given in [28].

Exercises

6.1. Consider $s, t \in \mathbb{Q}$. Show that

(a) $s \leq t$ if and only if $s^* \subseteq t^*$;

(b) $s = t$ if and only if $s^* = t^*$;

(c) $(s+t)^* = s^* + t^*$. Note that $s^* + t^*$ is a sum of Dedekind cuts.

6.2. Show that if α and β are Dedekind cuts, then so is $\alpha + \beta = \{r_1 + r_2 : r_1 \in \alpha \text{ and } r_2 \in \beta\}$.

6.3. **(a)** Show that $\alpha + 0^* = \alpha$ for all Dedekind cuts α.

(b) We claimed, without proof, that addition of Dedekind cuts satisfies property A4. Thus if α is a Dedekind cut, there must exist a Dedekind cut $-\alpha$ such that $\alpha + (-\alpha) = 0^*$. How would you define $-\alpha$?

6.4. Let α and β be Dedekind cuts and define the "product": $\alpha \cdot \beta = \{r_1 r_2 : r_1 \in \alpha \text{ and } r_2 \in \beta\}$.

(a) Calculate some "products" of Dedekind cuts using the Dedekind cuts 0^*, 1^* and $(-1)^*$.

(b) Discuss why this definition of "product" is totally unsatisfactory for defining multiplication in $\mathbb{R}$.

6.5. **(a)** Show that $\{r \in \mathbb{Q} : r^3 < 2\}$ is a Dedekind cut, but that $\{r \in \mathbb{Q} : r^2 < 2\}$ is not a Dedekind cut.

(b) Does the Dedekind cut $\{r \in \mathbb{Q} : r^3 < 2\}$ correspond to a rational number in $\mathbb{R}$?

(c) Show that $0^* \cup \{r \in \mathbb{Q} : r \geq 0 \text{ and } r^2 < 2\}$ is a Dedekind cut. Does it correspond to a rational number in $\mathbb{R}$?

6.6. Let $\alpha = 0^* \cup \{p \in \mathbb{Q} : p \geq 0 \text{ and } p^2 < 2\}$. Prove that α is a Dedekind cut and also that it has the property $\alpha \cdot \alpha = 2^*$; that is, the square of α is 2^*. *Note*: This seems to be surprisingly tricky, as pointed out by Linda Hill and Robert J. Fisher at Idaho State University. Their solution is available from them or from the author.

For $n > \max\{N_1, N_2\}$, the Triangle Inequality 3.7 shows that

$$|s - t| = |(s - s_n) + (s_n - t)| \le |s - s_n| + |s_n - t| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $|s - t| < \epsilon$ for all $\epsilon > 0$. It follows that $|s - t| = 0$, hence $s = t$.

Exercises

7.1. Write out the first five terms of the following sequences.

(a) $s_n = \frac{1}{3n+1}$ **(b)** $b_n = \frac{3n+1}{4n-1}$
(c) $c_n = \frac{n}{3^n}$ **(d)** $\sin(\frac{n\pi}{4})$

7.2. For each sequence in Exercise 7.1, determine whether it converges. If it converges, give its limit. No proofs are required.

7.3. For each sequence below, determine whether it converges and, if it converges, give its limit. No proofs are required.

(a) $a_n = \frac{n}{n+1}$ **(b)** $b_n = \frac{n^2+3}{n^2-3}$
(c) $c_n = 2^{-n}$ **(d)** $t_n = 1 + \frac{2}{n}$
(e) $x_n = 73 + (-1)^n$ **(f)** $s_n = (2)^{1/n}$
(g) $y_n = n!$ **(h)** $d_n = (-1)^n n$
(i) $\frac{(-1)^n}{n}$ **(j)** $\frac{7n^3+8n}{2n^3-31}$
(k) $\frac{9n^2-18}{6n+18}$ **(l)** $\sin(\frac{n\pi}{2})$
(m) $\sin(n\pi)$ **(n)** $\sin(\frac{2n\pi}{3})$
(o) $\frac{1}{n}\sin n$ **(p)** $\frac{2^{n+1}+5}{2^n-7}$
(q) $\frac{3^n}{n!}$ **(r)** $(1 + \frac{1}{n})^2$
(s) $\frac{4n^2+3}{3n^2-2}$ **(t)** $\frac{6n+4}{9n^2+7}$

7.4. Give examples of

(a) a sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.

(b) a sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

7.5. Determine the following limits. No proofs are required, but show any relevant algebra.

(a) $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$,

(b) $\lim(\sqrt{n^2+n} - n)$,

(c) $\lim(\sqrt{4n^2+n} - 2n)$.
Hint for (a): First show that $s_n = \frac{1}{\sqrt{n^2+1}+n}$.

§8 A Discussion about Proofs

In this section we give several examples of proofs using the definition of the limit of a sequence. With a little study and practice, students should be able to do proofs of this sort themselves. We will sometimes refer to a proof as a *formal proof* to emphasize that it is a rigorous mathematical proof.

Example 1
Prove that $\lim \frac{1}{n^2} = 0$.

Discussion. Our task is to consider an arbitrary $\epsilon > 0$ and show that there exists a number N [which will depend on ϵ] such that $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$. So we expect our formal proof to begin with "Let $\epsilon > 0$" and to end with something like "Hence $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$." In between the proof should specify an N and then verify that N has the desired property, namely that $n > N$ does indeed imply $|\frac{1}{n^2} - 0| < \epsilon$.

As is often the case with trigonometric identities, we will initially work backward from our desired conclusion, but in the formal proof we will have to be sure that our steps are reversible. In the present example, we want $|\frac{1}{n^2} - 0| < \epsilon$ and we want to know how big n must be. So we will operate on this inequality algebraically and try to "solve" for n. Thus we want $\frac{1}{n^2} < \epsilon$. By multiplying both sides by n^2 and dividing both sides by ϵ, we find that we want $\frac{1}{\epsilon} < n^2$ or $\frac{1}{\sqrt{\epsilon}} < n$. If our steps are reversible, we see that $n > \frac{1}{\sqrt{\epsilon}}$ implies $|\frac{1}{n^2} - 0| < \epsilon$. This suggests that we put $N = \frac{1}{\sqrt{\epsilon}}$.

Formal Proof
Let $\epsilon > 0$. Let $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ implies $n > \frac{1}{\sqrt{\epsilon}}$ which implies $n^2 > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^2}$. Thus $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$. This proves that $\lim \frac{1}{n^2} = 0$. ■

then we clearly have $m > 0$ and $|s_n| \geq m$ for *all* $n \in \mathbb{N}$ in view of (1). Thus $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$, as desired. ■

Formal proofs are required in the following exercises.

Exercises

8.1. Prove the following:
(a) $\lim \frac{(-1)^n}{n} = 0$ **(b)** $\lim \frac{1}{n^{1/3}} = 0$
(c) $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$ **(d)** $\lim \frac{n+6}{n^2-6} = 0$

8.2. Determine the limits of the following sequences, and then prove your claims.
(a) $a_n = \frac{n}{n^2+1}$ **(b)** $b_n = \frac{7n-19}{3n+7}$
(c) $c_n = \frac{4n+3}{7n-5}$ **(d)** $d_n = \frac{2n+4}{5n+2}$
(e) $s_n = \frac{1}{n} \sin n$

8.3. Let (s_n) be a sequence of nonnegative real numbers, and suppose that $\lim s_n = 0$. Prove that $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

8.4. Let (t_n) be a bounded sequence, i.e., there exists M such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.

8.5. **(a)** Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove that $\lim s_n = s$.

(b) Suppose that (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove that $\lim s_n = 0$.

8.6. Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove that $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

8.7. Show that the following sequences do not converge.
(a) $\cos(\frac{n\pi}{3})$ **(b)** $s_n = (-1)^n n$
(c) $\sin(\frac{n\pi}{3})$

8.8. Prove the following [see Exercise 7.5]:
(a) $\lim[\sqrt{n^2+1} - n] = 0$ **(b)** $\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$
(c) $\lim[\sqrt{4n^2+n} - 2n] = \frac{1}{4}$

8.9. Let (s_n) be

(a) Show t

(b) Show t

(c) Conclu
$\lim s_n$ l

8.10. Let (s_n) be
Prove that t

§9 Limit

In this section we
familiar to the re
bounded. A sequ
the set $\{s_n : n \in \mathbb{N}$
such that $|s_n| \leq M$

9.1 Theorem.
Convergent seque

Proof
Let (s_n) be a co
Definition 7.1 wi

From the triangle
Define $M = \max$
for all $n \in \mathbb{N}$, so (

In the proof of
for a single value

9.2 Theorem.
If the sequence $(s$
converges to ks. T

8.9. Let (s_n) be a sequence that converges.

   (a) Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

   (b) Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.

   (c) Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

8.10. Let (s_n) be a convergent sequence, and suppose that $\lim s_n > a$. Prove that there exists a number N such that $n > N$ implies $s_n > a$.

§9 Limit Theorems for Sequences

In this section we prove some basic results that are probably already familiar to the reader. First we prove that convergent sequences are bounded. A sequence (s_n) of real numbers is said to be *bounded* if the set $\{s_n : n \in \mathbf{N}\}$ is a bounded set, i.e., if there exists a constant M such that $|s_n| \leq M$ for all n.

9.1 Theorem.
Convergent sequences are bounded.

Proof
Let (s_n) be a convergent sequence, and let $s = \lim s_n$. Applying Definition 7.1 with $\epsilon = 1$ we obtain N in $\mathbf{N}$ so that

$$n > N \quad \text{implies} \quad |s_n - s| < 1.$$

From the triangle inequality we see that $n > N$ implies $|s_n| < |s| + 1$. Define $M = \max\{|s| + 1, |s_1|, |s_2|, \ldots, |s_N|\}$. Then we have $|s_n| \leq M$ for all $n \in \mathbf{N}$, so (s_n) is a bounded sequence. ■

In the proof of Theorem 9.1 we only needed to use property 7.1(1) for a single value of ϵ. Our choice of $\epsilon = 1$ was quite arbitrary.

9.2 Theorem.
If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then the sequence (ks_n) converges to ks. That is, $\lim(ks_n) = k \lim s_n$.

To prove (1), suppose that $\lim s_n = +\infty$. Let $\epsilon > 0$ and let $M = \frac{1}{\epsilon}$. Since $\lim s_n = +\infty$, there exists N such that $n > N$ implies $s_n > M = \frac{1}{\epsilon}$. Therefore $n > N$ implies $\epsilon > \frac{1}{s_n} > 0$, so

$$n > N \quad \text{implies} \quad \left|\frac{1}{s_n} - 0\right| < \epsilon.$$

That is, $\lim(\frac{1}{s_n}) = 0$. This proves (1).

To prove (2), we abandon the notation of the last paragraph and begin anew. Suppose that $\lim(\frac{1}{s_n}) = 0$. Let $M > 0$ and let $\epsilon = \frac{1}{M}$. Then $\epsilon > 0$, so there exists N such that $n > N$ implies $|\frac{1}{s_n} - 0| < \epsilon = \frac{1}{M}$. Since $s_n > 0$, we can write

$$n > N \quad \text{implies} \quad 0 < \frac{1}{s_n} < \frac{1}{M}$$

and hence

$$n > N \quad \text{implies} \quad M < s_n.$$

That is, $\lim s_n = +\infty$ and (2) holds. ■

Exercises

9.1. Using the limit theorems 9.2–9.6 and 9.7, prove the following. Justify all steps.

(a) $\lim \frac{n+1}{n} = 1$ **(b)** $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$

(c) $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$

9.2. Suppose that $\lim x_n = 3$, $\lim y_n = 7$ and that all y_n are nonzero. Determine the following limits:

(a) $\lim(x_n + y_n)$ **(b)** $\lim \frac{3y_n - x_n}{y_n^2}$

9.3. Suppose that $\lim a_n = a$, $\lim b_n = b$, and that $s_n = \frac{a_n^3+4a_n}{b_n^2+1}$. Prove $\lim s_n = \frac{a^3+4a}{b^2+1}$ carefully, using the limit theorems.

9.4. Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) List the first four terms of (s_n).

(b) It turns out that (s_n) converges. Assume this fact and prove that the limit is $\frac{1}{2}(1 + \sqrt{5})$.

9.5. Let $t_1 = 1$ and $t_{n+1} = \frac{t_n^2+2}{2t_n}$ for $n \geq 1$. Assume that (t_n) converges and find the limit.

9.6. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.

(a) Show that if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.

(b) Does $\lim x_n$ exist? Explain.

(c) Discuss the apparent contradiction between parts (a) and (b).

9.7. Complete the proof of 9.7(c), i.e., give the standard argument needed to show that $\lim s_n = 0$.

9.8. Give the following when they exist. Otherwise assert "NOT EXIST."

(a) $\lim n^3$ **(b)** $\lim(-n^3)$
(c) $\lim(-n)^n$ **(d)** $\lim(1.01)^n$
(e) $\lim n^n$

9.9. Suppose that there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.

(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

(b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

9.10. **(a)** Show that if $\lim s_n = +\infty$ and $k > 0$, then $\lim(ks_n) = +\infty$.

(b) Show that $\lim s_n = +\infty$ if and only if $\lim(-s_n) = -\infty$.

(c) Show that if $\lim s_n = +\infty$ and $k < 0$, then $\lim(ks_n) = -\infty$.

9.11. **(a)** Show that if $\lim s_n = +\infty$ and $\inf\{t_n : n \in \mathbf{N}\} > -\infty$, then $\lim(s_n + t_n) = +\infty$.

(b) Show that if $\lim s_n = +\infty$ and $\lim t_n > -\infty$, then $\lim(s_n + t_n) = +\infty$.

(c) Show that if $\lim s_n = +\infty$ and if (t_n) is a bounded sequence, then $\lim(s_n + t_n) = +\infty$.

9.12. Assume all $s_n \neq 0$ and that the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$. *Hint*: Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show that $|s_n| < a^{n-N}|s_N|$ for $n > N$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$. *Hint*: Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

9.13. Show that

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ 1 & \text{if } a = 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a \le -1. \end{cases}$$

9.14. Let $p > 0$. Use Exercise 9.12 to show

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \le 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a < -1. \end{cases}$$

9.15. Show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

9.16. Use Theorems 9.9, 9.10 or Exercises 9.9–9.15 to prove the following:

(a) $\lim \frac{n^4+8n}{n^2+9} = +\infty$

(b) $\lim[\frac{2^n}{n^2} + (-1)^n] = +\infty$

(c) $\lim[\frac{3^n}{n^3} - \frac{3^n}{n!}] = +\infty$

9.17. Give a formal proof that $\lim n^2 = +\infty$ using only Definition 9.8.

9.18. **(a)** Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \ne 1$.

(b) Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

(c) Calculate $\lim_{n\to\infty}(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n})$.

(d) What is $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $a \ge 1$?

§10 Monotone Sequences and Cauchy Sequences

In this section we obtain two theorems [Theorems 10.2 and 10.11] that will allow us to conclude that certain sequences converge *without* knowing the limit in advance. These theorems are important because in practice the limits are not usually known in advance.

10.1 Definition.
A sequence (s_n) of real numbers is called a *nondecreasing sequence* if $s_n \le s_{n+1}$ for all n, and (s_n) is called a *nonincreasing sequence*

The proof of Theorem 10.11 uses Theorem 10.7, and Theorem 10.7 relies implicitly on the Completeness Axiom 4.4, since without the completeness axiom it is not clear that $\liminf s_n$ and $\limsup s_n$ are meaningful. The completeness axiom assures us that the expressions $\sup\{s_n : n > N\}$ and $\inf\{s_n : n > N\}$ in Definition 10.6 are meaningful, and Theorem 10.2 [which itself relies on the completeness axiom] assures us that the limits in Definition 10.6 also are meaningful.

Exercises on lim sup's and lim inf's appear in §§11 and 12.

Exercises

10.1. Which of the following sequences are nondecreasing? nonincreasing? bounded?

(a) $\frac{1}{n}$ **(b)** $\frac{(-1)^n}{n^2}$

(c) n^5 **(d)** $\sin(\frac{n\pi}{7})$

(e) $(-2)^n$ **(f)** $\frac{n}{3^n}$

10.2. Prove Theorem 10.2 for bounded nonincreasing sequences.

10.3. For a decimal expansion $k.d_1d_2d_3d_4\cdots$, let (s_n) be defined as in 10.3. Prove that $s_n < k+1$ for all $n \in \mathbb{N}$. *Hint*: $\frac{9}{10}+\frac{9}{10^2}+\cdots+\frac{9}{10^n} = 1-\frac{1}{10^n}$ for all n.

10.4. Discuss why Theorems 10.2 and 10.11 would fail if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers.

10.5. Prove Theorem 10.4(ii).

10.6. **(a)** Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove that (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume that $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

10.7. Let S be a bounded nonempty subset of $\mathbb{R}$ and suppose $\sup S \notin S$. Prove that there is a nondecreasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.

10.8. Let (s_n) be a nondecreasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove that (σ_n) is a nondecreasing sequence.

10.9. Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

(b) Show that $\lim s_n$ exists.

(c) Prove that $\lim s_n = 0$.

10.10. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

(b) Use induction to show that $s_n > \frac{1}{2}$ for all n.

(c) Show that (s_n) is a nonincreasing sequence.

(d) Show that $\lim s_n$ exists and find $\lim s_n$.

10.11. Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{4n^2}] \cdot t_n$ for $n \geq 1$.

(a) Show that $\lim t_n$ exists.

(b) What do you think $\lim t_n$ is?

10.12. Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{(n+1)^2}] \cdot t_n$ for $n \geq 1$.

(a) Show that $\lim t_n$ exists.

(b) What do you think $\lim t_n$ is?

(c) Use induction to show that $t_n = \frac{n+1}{2n}$.

(d) Repeat part (b).

§11 Subsequences

11.1 Definition.
Suppose that $(s_n)_{n \in \mathbf{N}}$ is a sequence. A *subsequence* of this sequence is a sequence of the form $(t_k)_{k \in \mathbf{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \tag{1}$$

that $k > N$ implies $|t_k - t| < \frac{\epsilon}{2}$. If $k > \max\{N, \frac{2}{\epsilon}\}$, then $\frac{1}{k} < \frac{\epsilon}{2}$ and $|t_k - t| < \frac{\epsilon}{2}$, so $|s_{n_k} - t| < \epsilon$ by (3).]

Suppose next that $t = +\infty$. From (2) we have

$$s_{n_k} > t_k - \frac{1}{k} \quad \text{for} \quad k \in \mathbb{N}. \tag{4}$$

Since $\lim t_k = +\infty$ it follows easily that $\lim_{k\to\infty} s_{n_k} = +\infty$. Therefore $t = +\infty$ belongs to S. The case $t = -\infty$ is handled in a similar way. ■

Exercises

11.1. Let $a_n = 3 + 2(-1)^n$ for $n \in \mathbb{N}$.

- **(a)** List the first eight terms of the sequence (a_n).
- **(b)** Give a subsequence that is constant [takes a single value]. Specify the selection function σ.

11.2. Consider the sequences defined as follows:

$$a_n = (-1)^n, \qquad b_n = \frac{1}{n}, \qquad c_n = n^2, \qquad d_n = \frac{6n+4}{7n-3}.$$

- **(a)** For each sequence, give an example of a monotone subsequence.
- **(b)** For each sequence, give its set of subsequential limits.
- **(c)** For each sequence, give its lim sup and lim inf.
- **(d)** Which of the sequences converges? diverges to $+\infty$? diverges to $-\infty$?
- **(e)** Which of the sequences is bounded?

11.3. Repeat Exercise 11.2 for the sequences:

$$s_n = \cos(\frac{n\pi}{3}), \qquad t_n = \frac{3}{4n+1}, \qquad u_n = (-\frac{1}{2})^n, \qquad v_n = (-1)^n + \frac{1}{n}.$$

11.4. Repeat Exercise 11.2 for the sequences:

$$w_n = (-2)^n, \qquad x_n = 5^{(-1)^n}, \qquad y_n = 1 + (-1)^n, \qquad z_n = n\cos(\frac{n\pi}{4}).$$

11.5. Let (q_n) be an enumeration of all the rationals in the interval $(0, 1]$.

(a) Give the set of subsequential limits for (q_n).

(b) Give the values of $\limsup q_n$ and $\liminf q_n$.

11.6. Show that every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence. *Hint*: Define subsequences as in (3) of Definition 11.1.

11.7. Let (r_n) be an enumeration of the set $\mathbb{Q}$ of all rational numbers. Show that there exists a subsequence (r_{n_k}) such that $\lim_{k\to\infty} r_{n_k} = +\infty$.

11.8. **(a)** Use Definition 10.6 and Exercise 5.4 to prove that $\liminf s_n = -\limsup(-s_n)$.

(b) Let (t_k) be a monotonic subsequence of $(-s_n)$ converging to $\limsup(-s_n)$. Show that $(-t_k)$ is a monotonic subsequence of (s_n) converging to $\liminf s_n$. Observe that this completes the proof of Corollary 11.4.

11.9. **(a)** Show that the closed interval $[a, b]$ is a closed set.

(b) Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

11.10. Let (s_n) be the sequence of numbers in Figure 11.2 listed in the indicated order.

(a) Find the set S of subsequential limits of (s_n).

(b) Determine $\limsup s_n$ and $\liminf s_n$.

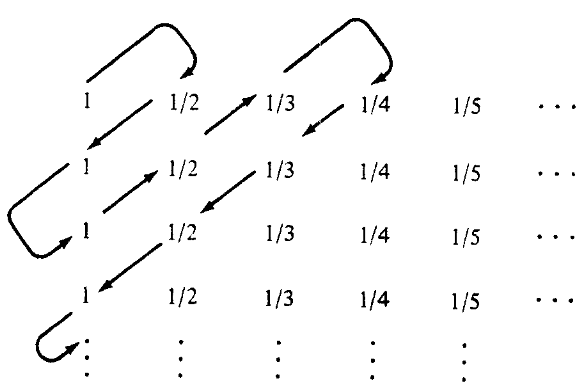

FIGURE 11.2

12.1. Let (s_n) and (t_n) be sequences and suppose that there exists N_0 such that $s_n \le t_n$ for all $n > N_0$. Show that $\liminf s_n \le \liminf t_n$ and $\limsup s_n \le \limsup t_n$. *Hint*: Use Definition 10.6 and Exercise 9.9(c).

12.2. Prove that $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

12.3. Let (s_n) and (t_n) be the following sequences that repeat in cycles of four:

$$(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \ldots)$$
$$(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, \ldots)$$

Find

(a) $\liminf s_n + \liminf t_n$, **(b)** $\liminf(s_n + t_n)$,
(c) $\liminf s_n + \limsup t_n$, **(d)** $\limsup(s_n + t_n)$,
(e) $\limsup s_n + \limsup t_n$, **(f)** $\liminf(s_n t_n)$,
(g) $\limsup(s_n t_n)$

12.4. Show that $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n). *Hint*: First show

$$\sup\{s_n + t_n : n > N\} \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Then apply Exercise 9.9(c).

12.5. Use Exercises 11.8(a) and 12.4 to prove

$$\liminf(s_n + t_n) \ge \liminf s_n + \liminf t_n$$

for bounded sequences (s_n) and (t_n).

12.6. Let (s_n) be a bounded sequence, and let k be a nonnegative real number.

(a) Prove that $\limsup(ks_n) = k \cdot \limsup s_n$.

(b) Do the same for $\liminf$. *Hint*: Use Exercise 11.8(a).

(c) What happens in (a) and (b) if $k < 0$?

12.7. Prove that if $\limsup s_n = +\infty$ and $k > 0$, then $\limsup(ks_n) = +\infty$.

12.8. Let (s_n) and (t_n) be bounded sequences of nonnegative numbers. Prove that $\limsup s_n t_n \le (\limsup s_n)(\limsup t_n)$.

12.9. **(a)** Prove that if $\lim s_n = +\infty$ and $\liminf t_n > 0$, then $\lim s_n t_n = +\infty$.

(b) Prove that if $\limsup s_n = +\infty$ and $\liminf t_n > 0$, then $\limsup s_n t_n = +\infty$.

(c) Observe that Exercise 12.7 is the special case of (b) where $t_n = k$ for all $n \in \mathbb{N}$.

12.10. Prove that (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.

12.11. Prove the first inequality in Theorem 12.2.

12.12. Let (s_n) be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

(a) Show that

$$\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n.$$

Hint: For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

12.13. Let (s_n) be a bounded sequence in $\mathbb{R}$. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e., all but finitely many s_n are $\ge a$. Let B be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove that $\sup A = \liminf s_n$ and $\inf B = \limsup s_n$.

12.14. Calculate (a) $\lim (n!)^{1/n}$, (b) $\lim \frac{1}{n}(n!)^{1/n}$.

§13 * Some Topological Concepts in Metric Spaces

In this book we are restricting our attention to analysis on $\mathbb{R}$. Accordingly, we have taken full advantage of the order properties of $\mathbb{R}$ and studied such important notions as lim sup's and lim inf's. In §3 we briefly introduced a distance function on $\mathbb{R}$. Most of our analysis could have been based on the notion of distance, in which case it becomes easy and natural to work in a more general setting. For example, analysis on the k-dimensional Euclidean spaces $\mathbb{R}^k$ is important, but these spaces do not have the useful natural ordering that $\mathbb{R}$ has, unless of course $k = 1$.

By Theorem 13.10, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains a point $\boldsymbol{x}_0$. This point belongs to some set U_0 in $\mathcal{U}$. Since U_0 is open, there exists $r > 0$ so that

$$\{\boldsymbol{x} \in \mathbb{R}^k : d(\boldsymbol{x}, \boldsymbol{x}_0) < r\} \subseteq U_0.$$

It follows that $F_n \subseteq U_0$ provided $\delta \cdot 2^{-n} < r$, but this contradicts (3) in a dramatic way. ■

Since $\mathbb{R} = \mathbb{R}^1$, the preceding results apply to $\mathbb{R}$.

Exercises

13.1. For points $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^k$, let

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_j - y_j| : j = 1, 2, \ldots, k\}$$

and

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{k} |x_j - y_j|.$$

(a) Show that d_1 and d_2 are metrics for $\mathbb{R}^k$.

(b) Show that d_1 and d_2 are complete.

13.2. **(a)** Prove (1) in Lemma 13.3.

(b) Prove the first assertion in Lemma 13.3.

13.3. Let B be the set of all bounded sequences $\boldsymbol{x} = (x_1, x_2, \ldots)$, and define $d(\boldsymbol{x}, \boldsymbol{y}) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\}$.

(a) Show that d is a metric for B.

(b) Does $d^*(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for B?

13.4. Prove (iii) and (iv) in Discussion 13.7.

13.5. **(a)** Verify one of DeMorgan's Laws for sets:

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}.$$

(b) Show that the intersection of any collection of closed sets is a closed set.

13.6. Prove Proposition 13.9.

13.7. Show that ev
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13.8. **(a)** Verify tl
(b) Verify tl

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Example 8

Consider the series

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \cdots. \qquad (1)$$

Let $a_n = 2^{(-1)^n - n}$. Since $a_n \le \frac{1}{2^{n-1}}$ for all n, we can quickly conclude that the series converges by the Comparison Test. But our real interest in this series is that it illustrates the difference between the Ratio Test and the Root Test. Since $a_{n+1}/a_n = 1/8$ for even n and $a_{n+1}/a_n = 2$ for odd n, we have

$$\frac{1}{8} = \liminf \left|\frac{a_{n+1}}{a_n}\right| < 1 < \limsup \left|\frac{a_{n+1}}{a_n}\right| = 2.$$

Hence the Ratio Test gives no information.

Note that $(a_n)^{1/n} = 2^{\frac{1}{n}-1}$ for even n and $(a_n)^{1/n} = 2^{-\frac{1}{n}-1}$ for odd n. Since $\lim 2^{\frac{1}{n}} = \lim 2^{-\frac{1}{n}} = 1$ by Example 7(d) in §9, we conclude that $\lim(a_n)^{1/n} = \frac{1}{2}$. Therefore $\alpha = \limsup(a_n)^{1/n} = \frac{1}{2} < 1$ and the series (1) converges by the Root Test.

Example 9

Consider the series

$$\sum \frac{(-1)^n}{\sqrt{n}}. \qquad (1)$$

Since $\lim \sqrt{n/(n+1)} = 1$, neither the Ratio Test nor the Root Test gives any information. Since $\sum \frac{1}{\sqrt{n}}$ diverges, we will not be able to use the Comparison Test 14.6(i) to show that (1) converges. Since the terms of the series (1) are not all nonnegative, we will not be able to use the Comparison Test 14.6(ii) to show that (1) diverges. It turns out that this series converges by the Alternating Series Test 15.3, which we have deferred to the next section.

Exercises

14.1. Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{n^4}{2^n}$ **(b)** $\sum \frac{2^n}{n!}$

(c) $\sum \frac{n^2}{3^n}$
(d) $\sum \frac{n!}{n^4+3}$
(e) $\sum \frac{\cos^2 n}{n^2}$
(f) $\sum_{n=2}^{\infty} \frac{1}{\log n}$

14.2. Repeat Exercise 14.1 for the following.

(a) $\sum \frac{n-1}{n^2}$
(b) $\sum (-1)^n$
(c) $\sum \frac{3n}{n^3}$
(d) $\sum \frac{n^3}{3^n}$
(e) $\sum \frac{n^2}{n!}$
(f) $\sum \frac{1}{n^n}$
(g) $\sum \frac{n}{2^n}$

14.3. Repeat Exercise 14.1 for the following.

(a) $\sum \frac{1}{\sqrt{n!}}$
(b) $\sum \frac{2+\cos n}{3^n}$
(c) $\sum \frac{1}{2^n+n}$
(d) $\sum (\frac{1}{2})^n (50 + \frac{2}{n})$
(e) $\sum \sin(\frac{n\pi}{9})$
(f) $\sum \frac{(100)^n}{n!}$

14.4. Repeat Exercise 14.1 for the following.

(a) $\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}$
(b) $\sum [\sqrt{n+1} - \sqrt{n}]$
(c) $\sum \frac{n!}{n^n}$

14.5. Suppose that $\sum a_n = A$ and $\sum b_n = B$ where A and B are real numbers. Use limit theorems from § 9 to quickly prove the following.

(a) $\sum (a_n + b_n) = A + B$.

(b) $\sum k a_n = kA$ for $k \in \mathbb{R}$.

(c) Is $\sum a_n b_n = AB$ a reasonable conjecture? Discuss.

14.6. (a) Prove that if $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges. *Hint*: Use Theorem 14.4.

(b) Observe that Corollary 14.7 is a special case of part (a).

14.7. Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

14.8. Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. *Hint*: Show that $\sqrt{a_n b_n} \le a_n + b_n$ for all n.

14.9. The convergence of a series does not depend on any finite number of the terms, though of course the value of the limit does. More precisely, consider series $\sum a_n$ and $\sum b_n$ and suppose that the set $\{n \in \mathbb{N} : a_n \ne b_n\}$ is finite. Then the series both converge or else they both diverge. Prove this. *Hint*: This is almost obvious from Theorem 14.4.

14.10. Find a series $\sum a_n$ which diverges by the Root Test but for which the Ratio Test gives no information. Compare Example 8.

14.11. Let (a_n) be a sequence of nonzero real numbers such that the sequence $(\frac{a_{n+1}}{a_n})$ of ratios is a constant sequence. Show that $\sum a_n$ is a geometric series.

14.12. Let $(a_n)_{n\in\mathbf{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove that there is a subsequence $(a_{n_k})_{k\in\mathbf{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

14.13. We have seen that it is often a lot harder to find the value of an infinite sum than to show that it exists. Here are some sums that can be handled.

(a) Calculate $\sum_{n=1}^{\infty}(\frac{2}{3})^n$ and $\sum_{n=1}^{\infty}(-\frac{2}{3})^n$.

(b) Prove $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$. *Hint*: Note that $\sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}[\frac{1}{k} - \frac{1}{k+1}]$.

(c) Prove that $\sum_{n=1}^{\infty}\frac{n-1}{2^{n+1}} = \frac{1}{2}$. *Hint*: Note that $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$.

(d) Use (c) to calculate $\sum_{n=1}^{\infty}\frac{n}{2^n}$.

14.14. Prove that $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges by comparing with the series $\sum_{n=2}^{\infty} a_n$ where (a_n) is the sequence

$$\left(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{32},\frac{1}{32},\ldots\right).$$

§15 Alternating Series and Integral Tests

Sometimes one can check convergence or divergence of series by comparing the partial sums with familiar integrals. We illustrate.

Example 1

We show that $\sum \frac{1}{n} = +\infty$.

Consider the picture of the function $f(x) = \frac{1}{x}$ in Figure 15.1. For $n \geq 1$ it is evident that

$$\sum_{k=1}^{n}\frac{1}{k} = \text{Sum of the areas of the first } n \text{ rectangles in Figure 15.1}$$

Remember that the numbers in brackets are nonnegative, since (a_n) is nonincreasing. If $n - m$ is even, the last term of A is $+a_n$, so

$$A = [a_m - a_{m+1}] + [a_{m+2} - a_{m+3}] + \cdots + [a_{n-2} - a_{n-1}] + a_n \geq 0$$

and

$$A = a_m - [a_{m+1} - a_{m+2}] - [a_{m+3} - a_{m+4}] - \cdots - [a_{n-1} - a_n] \leq a_m.$$

In either case we have $0 \leq A \leq a_m$. Hence from (2) we see that

$$\left|\sum_{k=m}^{n}(-1)^k a_k\right| = A \leq a_m.$$

Assertion (1) now follows since $n \geq m > N$ implies

$$\left|\sum_{k=m}^{n}(-1)^k a_k\right| \leq a_m \leq a_N.$$

■

Exercises

15.1. Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{(-1)^n}{n}$ **(b)** $\sum \frac{(-1)^n n!}{2^n}$

15.2. Repeat Exercise 15.1 for the following.

(a) $\sum[\sin(\frac{n\pi}{6})]^n$ **(b)** $\sum[\sin(\frac{n\pi}{7})]^n$

15.3. Show that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

15.4. Determine which of the following series converge. Justify your answers.

(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$ **(b)** $\sum_{n=2}^{\infty} \frac{\log n}{n}$

(c) $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log\log n)}$ **(d)** $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$

15.5. Why didn't we use the Comparison Test to prove Theorem 15.1 for $p > 1$?

15.6. **(a)** Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

(b) Observe that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges. See Exercise 14.7.

(c) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

15.7. **(a)** Prove that if (a_n) is a nonincreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim na_n = 0$. *Hint*: Consider $|a_{N+1} + a_{N+2} + \cdots + a_n|$ for suitable N.

(b) Use (a) to give another proof that $\sum \frac{1}{n}$ diverges.

15.8. Formulate and prove a general integral test as advised in 15.2.

§16 * Decimal Expansions of Real Numbers

We begin by recalling the brief discussion of decimals in Discussion 10.3. There we considered a decimal expansion $k.d_1d_2d_3\cdots$ where k is a nonnegative integer and each digit d_j belongs to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. This expansion represents the real number

$$k + \sum_{j=1}^{\infty} \frac{d_j}{10^j} = k + \sum_{j=1}^{\infty} d_j \cdot 10^{-j}$$

which we also can write as

$$\lim_{n\to\infty} s_n \quad \text{where} \quad s_n = k + \sum_{j=1}^{n} d_j \cdot 10^{-j}.$$

Thus *every such decimal expansion represents a nonnegative real number*. We will prove the converse after we formalize the process of long division. The development here is based on some suggestions by Karl Stromberg.

16.1 Long Division.
Let's first consider positive integers a and b where $a < b$. We analyze the familiar long division process which gives a decimal expansion for $\frac{a}{b}$. Figure 16.1 shows the first few steps where $a = 3$ and $b = 7$. If we name the digits $d_1, d_2, d_3, \ldots$ and the remainders $r_1, r_2, r_3, \ldots$, then so far $d_1 = 4$, $d_2 = 2$ and $r_1 = 2$, $r_2 = 6$. At the next step we divide 7 into $60 = 10 \cdot r_2$ and obtain $60 = 7 \cdot 8 + 4$. The quotient 8 becomes the

the proof that

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

is irrational. Assume that $e = \frac{a}{b}$ where $a, b \in \mathbb{N}$. Then both $b!e$ and $b! \sum_{k=0}^{b} \frac{1}{k!}$ must be integers, so the difference

$$b! \sum_{k=b+1}^{\infty} \frac{1}{k!}$$

must be a positive integer. On the other hand, this last number is less than

$$\frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots = \frac{1}{b} \le 1,$$

a contradiction.

Example 7

There is a famous number introduced by Euler over 200 years ago that arises in the study of the gamma function. It is known as *Euler's constant* and is defined by

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log_e n \right].$$

Even though

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = +\infty \quad \text{and} \quad \lim_{n\to\infty} \log_e n = +\infty,$$

the limit defining γ exists and is finite [Exercise 16.9]. In fact, γ is approximately .577216. The amazing fact is that no one knows whether γ is rational or not. Most mathematicians believe γ is irrational. This is because it is "easier" for a number to be irrational, since repeating decimal expansions must be regular. The remark in Exercise 16.8 hints at another reason it is easier for a number to be irrational.

Exercises

16.1. (a) Show that $2.749\overline{9}$ and $2.75\overline{0}$ are both decimal expansions for $\frac{11}{4}$.

(b) Which of these expansions arises from the long division process described in 16.1?

16.2. Verify the claims in the first paragraph of the proof of Theorem 16.3.

16.3. Suppose that $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers. Show that if $a_n \le b_n$ for all n and if $a_n < b_n$ for at least one n, then $\sum a_n < \sum b_n$.

16.4. Write the following repeating decimals as rationals, i.e., as fractions of integers.

(a) $.\overline{2}$ **(b)** $.0\overline{2}$
(c) $.\overline{02}$ **(d)** $3.\overline{14}$
(e) $.\overline{10}$ **(f)** $.1\overline{492}$

16.5. Find the decimal expansions of the following rational numbers.

(a) $\frac{1}{8}$ **(b)** $\frac{1}{16}$
(c) $\frac{2}{3}$ **(d)** $\frac{7}{9}$
(e) $\frac{6}{11}$ **(f)** $\frac{22}{7}$

16.6. Find the decimal expansions of $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$. Note the interesting pattern.

16.7. Is $.12345678910111213141516171819202122232425 26\cdots$ rational?

16.8. Let (s_n) be a sequence of numbers in $(0,1)$. Each s_n has a decimal expansion $.d_1^{(n)}d_2^{(n)}d_3^{(n)}\cdots$. For each n, let $e_n = 6$ if $d_n^{(n)} \ne 6$ and $e_n = 7$ if $d_n^{(n)} = 6$. Show that $e_1e_2e_3\cdots$ is the decimal expansion for some number y in $(0,1)$ and that $y \ne s_n$ for all n. *Remark*: This shows that the elements of $(0,1)$ cannot be listed as a sequence. In set-theoretic parlance, $(0,1)$ is "uncountable." Since the set $\mathbb{Q}\cap(0,1)$ can be listed as a sequence, there must be a lot of irrational numbers in $(0,1)$!

16.9. Let $\gamma_n = \left(\sum_{k=1}^{n}\frac{1}{k}\right) - \log_e n = \sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{1}{t}\,dt$.

(a) Show that (γ_n) is a decreasing sequence. *Hint*: Look at $\gamma_n - \gamma_{n+1}$.

(b) Show that $0 < \gamma_n \le 1$ for all n.

(c) Observe that $\gamma = \lim_n \gamma_n$ exists and is finite.

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§17 Cont

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