Name: Solutions

Student ID#: ________________

Section: ________________

Midterm Exam 1

MAT 25–Temple

Wednesday, February 8, 2017

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Short Answer: (**a**) Find the Least Upper Bound and Greatest Lower Bound of the set

$$S = \left\{ (-1)^n - \frac{1}{n^2} : n \in \mathbf{N} \right\}.$$

LUB = 1

GLB = -2

(**b**) Let a and b be real numbers. Use the triangle inequality to bound $|a+b|$ *above* and *below* in terms of $|a|$ and $|b|$. (You need not justify your answer.)

$$\big| |a| - |b| \big| \leq |a+b| \leq |a| + |b|$$

(**c**) Is the following statement true or false? "If $\sqrt{-1} > 0$, then *every real number is rational.*" Justify your answer.

Statement True

Reason: If P then Q is true if P = F & Q = F

(**d**) Without using the negation sign or the word "not", write the negation of the following sentence employing the *For Every* and *There Exists* symbols:

There exists $s_0 \in \mathbf{R}$ such that for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that for every $n > N$, $|s_n - s_0| < \epsilon$.

$$\neg \exists s_0 \in \mathbb{R}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ |S_n - S_0| < \varepsilon$$

$$\Leftrightarrow \forall s_0 \in \mathbb{R}\ \exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n > N\ \underbrace{\neg |S_n - S_0| < \varepsilon}_{|S_n - S_0| \geq \varepsilon}$$

Problem #2 (20pts): Using only the field axioms for the real numbers, prove that $a \cdot 0 = 0$ for every real number $a \in \mathbf{R}$. Justify every step.

$a \cdot 0 = a \cdot (0+0)$ (add identity)

$= a \cdot 0 + a \cdot 0$ (dist prop)

$a \cdot 0 - a \cdot 0 = a \cdot 0 + a \cdot 0 - a \cdot 0$ (did same thing to both sides of = sign)

$0 = a \cdot 0$ (defn add inverse)

Problem #3 (20pts): Using the definition, prove that if $s_n \to s_0$ and $t_n \to t_0$ then $s_n t_n \to s_0 t_0$.

We prove: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ st $n > N \Rightarrow |s_n t_n - s_0 t_0| < \varepsilon$

So fix $\varepsilon > 0$. We find N st $n > N \Rightarrow |s_n t_n - s_0 t_0| < \varepsilon$

$$|s_n t_n - s_0 t_0| = |s_n t_n - s_0 t_n + s_0 t_n - s_0 t_0|$$

$$\le \underbrace{|t_n||s_n - s_0|}_{< \frac{\varepsilon}{2}} + \underbrace{|s_0||t_n - t_0|}_{< \frac{\varepsilon}{2}}$$

Choose N_1 st $n > N_1 \Rightarrow |t_n - t_0| < \frac{\varepsilon}{2|s_0|}$ (wlog $s_0 \neq 0$)

Choose N_2 st $n > N_2 \Rightarrow \cancel{M}|s_n - s_0| < \frac{\varepsilon}{2M}$

where $M > |t_n|$ is the bound for t_n

Choose $N = \max\{N_1, N_2\}$. Then $n > N \Rightarrow$

$$|s_n t_n - s_0 t_0| \le |t_n||s_n - s_0| + |s_0||t_n - t_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \checkmark$$

Problem #4 (20pts): Let r be a real number such that $|r| < 1$, and let s_n denote the sequence of partial sum $s_n = \sum_{k=2}^{n} r^k = r^2 + r^3 + \cdots + r^n$. Derive a formula for s_n that does not involve a summation, and use it to evaluate $\lim_{n\to\infty} s_n$.

$$S_n = r^2 + r^3 + \cdots + r^n$$

$$rS_n = r^3 + r^4 + \cdots + r^{n+1}$$

$$S_n - rS_n = r^2 - r^{n+1}$$

$$(1-r)S_n = r^2 - r^{n+1}$$

$$S_n = \frac{r^2 - r^{n+1}}{1-r} \longrightarrow \frac{r^2}{1-r}$$

Problem #5 (20pts): Assume that $0 < s_n < 2t_n$ and that $t_n \to 0$. Use the definition of convergence to give a careful proof that $s_n \to 0$.

We prove: $\forall \varepsilon \exists N$ st $n > N \Rightarrow |s_n - 0| < \varepsilon$

Fix $\varepsilon > 0$. We find N st $n > N \Rightarrow |s_n| < \varepsilon$

But $|s_n| < 2|t_n|$ and $t_n \to 0$, so choose N st $n > N \Rightarrow |t_n| < \frac{\varepsilon}{2}$. Then $n > N \Rightarrow$

$$|s_n| < 2|t_n| < 2 \cdot \frac{\varepsilon}{2} < \varepsilon$$