Name: Solutions

Student ID#:

Section:

Midterm Exam 2

MAT 25–Temple

Wednesday, March 1, 2017

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

6pt **Problem #1 (20pts): Short Answer:** (**a**) Define what it means for a sequence of real numbers s_n to be Cauchy. Then state the Cauchy Equivalence of Convergence Theorem.

s_n Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ st $n,m > N \Rightarrow |s_n - s_m| < \varepsilon$

Thm: a sequence s_n converges to real number iff s_n is Cauchy

8pt (**b**) Define $\underline{s}_N$, $\bar{s}_N$, liminf(s_n), and limsup(s_n). 2 each

$$\underline{s}_N = GLB\{s_n : n > N\} \quad \underline{s} = \lim_{N\to\infty} \underline{s}_N$$

$$\bar{s}_N = LUB\{s_n : n > N\} \quad \bar{s} = \lim_{N\to\infty} \bar{s}_N$$

6pt (**c**) Write the correct inequalities that hold between $\underline{s}_N$, $\bar{s}_N$, liminf(s_n) and limsup(s_n). (You need not justify your answers.)

$$\underline{s}_N \leq \underline{s} \leq \tilde{s} \leq \tilde{s}_N$$

2pt 2pts 2pts

Problem #2 (20pts): Short Answer:

(**a**) A sequence is convergent if:

$$\exists\ s_0 \in \mathcal{R}\ st\ \forall \epsilon > 0\ \exists\ N \in \mathcal{N}\ st\ \forall\ n > N,\ |s_n - s_0| < \epsilon.$$

Write the statement which asserts the sequence does *not* converge.

$\forall\ s_0 \in \mathbb{R}\ \exists\ \varepsilon > 0\ st\ \forall\ N \in \mathbb{N}\ \exists\ n > N\ |s_n - s_0| \geq \epsilon$

(**b**) State the Bolzano-Weierstrass Theorem:

Every bounded sequence s_n contains a convergent subsequence

(**c**) Define what it means for a set to be (sequentially) *closed*.

S is sequentially closed if it contains the limits of all convergent subsequences

Problem #3 (20pts): (a) Give the definition of $s_n \to -\infty$.

$S_n \to -\infty$ means: $\forall M > 0\ \exists N \in \mathbb{N}$ st
$n > N \Rightarrow S_n < -M$.

8pts

12 pts (b) Prove directly that if $s_n \to -\infty$, then every subsequence $s_{n_k} \to -\infty$.

Assume $S_n \to -\infty$, and let S_{n_k} be a subsequen of S_n.

We prove $S_{n_k} \to -\infty$.

So fix $M > 0$. We find N st $k > N \Rightarrow S_{n_k} < -M$

But we know $S_n \to -\infty$, so $\exists\ N_1$ st $n > N_1$ implies $S_n < -M$. Now $n_k \to \infty$ as $k \to \infty$,

So choose N st $k > N$ implies $n_k > N_1$.

Then $k > N$ implies $S_{n_k} < -M$ because $n_k > N_1$.

Problem #4 (20pts): Assume $s_n \to s_0$ converges to a real number s_0, and let t_n be a bounded sequence. Prove

$$\text{limsup}\,(s_n t_n) = s_0\,\text{limsup}\,(t_n).$$

Let $B = \limsup t_n$. We prove that $\forall \varepsilon > 0$,

$$\left| \limsup s_n t_n - s_0 B \right| \leq \varepsilon.$$

Choose N st $n > N \Rightarrow |s_n - s_0| < \frac{\varepsilon}{M}$, where $|t_n| < M$

Then $|s_n t_n - s_0 t_n| = |s_0 - s_n||t_n| < \varepsilon$ for $n > N$.

But if two sequences a_n & b_n satisfy $|a_n - b_n| < \varepsilon$, then $|\limsup a_n - \limsup b_n| \leq \varepsilon$.

To see this, $\bar{a}_N = LUB\{a_n : n > N\}$

$\bar{b}_N = LUB\{b_n : n > N\}$

and since $|a_n - b_n| < \varepsilon$, also $|\bar{a}_N - \bar{b}_N| < \varepsilon$, so

$\lim_{N \to \infty} |\bar{a}_N - \bar{b}_N| = |\bar{a} - \bar{b}| \leq \varepsilon$ ✓. But then $n > N$

we have $|s_n t_n - s_0 t_0| < \varepsilon$. $\Rightarrow |\limsup s_n t_n - \limsup s_0 t_n| \leq \varepsilon$

$\Rightarrow |\limsup s_n t_n - s_0 \limsup t_n| \leq \varepsilon$ as claimed

Problem #5 (20pts): Let s_n be a bounded sequence. We call a term s_N in the sequence *diminutive* if it is smaller than all the terms in the sequence which follows it, i.e., $s_N \leq s_n$ for all $n \geq N$. Prove directly that if a sequence has only a finite number of diminutive terms, then it contains a monotone subsequence.

Since there are only finitely many diminutive terms, choose N such that $\nexists$ diminutive terms for $n > N$. Choose S_{n_1} with $n_1 > N$.
Now n_1 not diminutive implies $\exists\ n_2 > n_1$ such that $S_{n_2} \leq S_{n_1}$. (otherwise, S_{n_1} is diminutive)
Similarly, S_{n_2} not diminutive implies $\exists$ $n_3 > n_2$ such that $S_{n_3} \leq S_{n_2}$. Continuing to n_k, there exists $n_{k+1} > n_k$ such that $S_{n_{k+1}} \leq S_{n_k}$ because S_{n_k} is not diminutive.
Thus by induction, $S_{n_{k+1}} \leq S_{n_k} \Rightarrow \{S_{n_k}\}$ is a monotone decreasing sequence. ✓