Midterm Exam 2  
MAT 25–Temple  
Wednesday, March 1, 2017

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

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Problem #1 (20pts): Short Answer: (a) Define what it means for a sequence of real numbers $s_n$ to be Cauchy. Then state the Cauchy Equivalence of Convergence Theorem.

\[ s_n \text{ Cauchy if } \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } m, n > N \implies |s_m - s_n| < \epsilon \]

Thm: a sequence $s_n$ converges to real number iff $s_n$ is Cauchy

(b) Define $\underline{s}_N$, $\overline{s}_N$, $\liminf(s_n)$, and $\limsup(s_n)$.

\[
\underline{s}_N = \text{GLB} \{s_n : n > N\} \quad s = \lim_{n \to \infty} s_N \\
\overline{s}_N = \text{LUB} \{s_n : n > N\} \quad \bar{s} = \lim_{n \to \infty} \overline{s}_N
\]

(c) Write the correct inequalities that hold between $\underline{s}_N$, $\overline{s}_N$, $\liminf(s_n)$ and $\limsup(s_n)$. (You need not justify your answers.)

\[
\underline{s}_N \leq \underline{s} \leq \bar{s} \leq \overline{s}_N
\]

2 pts 2 pts 2 pts
Problem #2 (20pts): Short Answer:
(a) A sequence is convergent if:

\[ \exists s_0 \in \mathbb{R} \text{ st } \forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ st } \forall n > N, \ |s_n - s_0| < \varepsilon. \]

Write the statement which asserts the sequence does not converge.

\[ \forall s_0 \in \mathbb{R} \exists \varepsilon > 0 \forall N \in \mathbb{N} \exists n > N \text{ st } |s_n - s_0| \geq \varepsilon \]

(b) State the Bolzano-Weierstrass Theorem:

Every bounded sequence \( s_n \) contains a convergent subsequence.

c) Define what it means for a set to be (sequentially) closed.

\( S \) is sequentially closed if it contains the limits of all convergent subsequences.
Problem #3 (20pts): (a) Give the definition of $s_n \to -\infty$.

$$s_n \to -\infty \text{ means: } \forall M > 0 \exists N \in \mathbb{N} \text{ s.t. } n > N \Rightarrow s_n < -M.$$ 8pts

(b) Prove directly that if $s_n \to -\infty$, then every subsequence $s_{n_k} \to -\infty$.

Assume $s_n \to -\infty$, and let $s_{n_k}$ be a subsequence of $s_n$. We prove $s_{n_k} \to -\infty$.

So fix $M > 0$. We find $N$ s.t. $k > N \Rightarrow s_{n_k} < -M$.

But we know $s_n \to -\infty$, so $\exists N_1$ s.t. $n > N_1$ implies $s_n < -M$. Now $n_k \to \infty$ as $k \to \infty$.

So choose $N$ s.t. $k > N$ implies $n_k > N_1$. Then $k > N$ implies $s_{n_k} < -M$ because $n_k > N$.

12pts
Problem #4 (20pts): Assume $s_n \to s_0$ converges to a real number $s_0$, and let $t_n$ be a bounded sequence. Prove
\[ \limsup (s_n t_n) = s_0 \limsup (t_n). \]

Let $B = \limsup t_n$. We prove that $\forall \varepsilon > 0$,
\[ \left| \limsup s_n t_n - s_0 B \right| \leq \varepsilon. \]
Choose $N$ such that $n > N \implies |s_n - s_0| < \frac{\varepsilon}{M}$, where $|t_n| < M$.

Then $|s_n t_n - s_0 t_n| = |s_0 - s_n| |t_n| < \varepsilon$ for $n > N$.

But if two sequences $a_n$ and $b_n$ satisfy $|a_n - b_n| < \varepsilon$, then $\left| \limsup a_n - \limsup b_n \right| \leq \varepsilon$.

To see this, \( \bar{a}_N = \mathrm{LUB} \{a_n : n > N\} \)
\( \bar{b}_N = \mathrm{LUB} \{b_n : n > N\} \)
and since $|a_n - b_n| < \varepsilon$, also $|\bar{a}_N - \bar{b}_N| < \varepsilon$, so
\[ \lim \left| \bar{a}_N - \bar{b}_N \right| = |\bar{a} - \bar{b}| \leq \varepsilon \vee. \]

But then $n > N$ we have $|s_n t_n - s_0 t_n| < \varepsilon \iff \limsup s_n t_n - \limsup s_0 t_n \leq \varepsilon$
\[ \implies \limsup s_n t_n - s_0 \limsup t_n \leq \varepsilon \] as claimed.
Problem #5 (20pts): Let $s_n$ be a bounded sequence. We call a term $s_N$ in the sequence *diminutive* if it is smaller than all the terms in the sequence which follows it, i.e., $s_N \leq s_n$ for all $n \geq N$. Prove directly that if a sequence has only a finite number of diminutive terms, then it contains a monotone subsequence.

Since there are only finitely many diminutive terms, choose $N$ such that $\exists$ diminutive terms for $n > N$. Choose $s_{n_1}$ with $n_1 > N$. Now $\neg$ not diminutive implies $\exists n_2 > n_1$ such that $s_{n_2} \leq s_{n_1}$. (Otherwise, $s_{n_1}$ is diminutive.)

Similarly, $s_{n_2}$ not diminutive implies $\exists n_3 > n_2$ such that $s_{n_3} \leq s_{n_2}$. Continuing to $n_k$, there exists $n_{k+1} > n_k$ such that $s_{n_{k+1}} \leq s_{n_k}$ because $s_{n_k}$ is not diminutive.

Thus by induction, $s_{n_{k+1}} \leq s_{n_k} \Rightarrow \{s_{n_k}\}$ is a monotone decreasing sequence.