MIDTERM EXAM II
Math 25
Temple-F06

Write solutions on the paper provided. Put your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

Problem 1. (a) (4pts) Give the precise definition of an upper bound for a set $A \subseteq \mathbb{R}$.

(b) (4pts) Give the precise definition of the least upper bound for a set $A \subseteq \mathbb{R}$.

(c) (4pts) State the Completeness Axiom regarding a $A \subseteq \mathbb{R}$.

(d) (8pts) Let $s_n$ be a non-decreasing sequence of real numbers that is bounded from above. Give a careful proof that the sequence $s_n$ converges.

Problem 2. Let $s_n$ be a sequence of real numbers.

(a) (2pts) Give the precise definition for $s_n \to s_0$ in the case when $s_0 \in \mathbb{R}$, and when $s_0 = +\infty$.

(b) (4pts) Define what it means for $s_n$ to be a Cauchy sequence.

(c) (4pts) Give the negation of the statement “$s_n$ is Cauchy”.

(d) (5pts) Prove directly: If $s_n$ converges to a real number $s_0$, then $s_n$ is Cauchy.

(e) (5pts) Prove directly: If $s_n \to +\infty$, then $s_n$ is not Cauchy.
**Problem 3.** Let $s_n$ be a sequence of real numbers.

(a) (6pts) Define $s_N$ and $\bar{s}_N$, the approximate lim inf and approximate lim sup of $s_n$, respectively.

(b) (6pts) Define $\underline{s} = \lim\inf s_n$ and $\bar{s} = \lim\sup s_n$ in terms of $s_N$ and $\bar{s}_N$, respectively.

(c) (8pts) Use $\leq$ to give the correct inequalities that order the set $\{s_N, \bar{s}_N, \underline{s}, \bar{s}\}$, and prove the right most inequality.

**Problem 4.** Let $s_n = \{1 + (-1)^n\} e^{1/n}$

(a) (10pts) Find the lim inf $s_n$ and lim sup $s_n$.

(b) (10pts) Define a subsequence that converges to lim sup $s_n$.

**Problem 5.** Let $s_n$ be a sequence of real numbers.

(a) (10pts) Give an example of a sequence of real numbers $s_n$ whose subsequential limit set is exactly the set $\{1/n : n \in \mathbb{N}\} \cup \{0\}$. (You may use a diagram to define your sequence.)

(b) (10pts) Prove that there is no sequence of real numbers whose subsequential limit set $S$ is the set $\{1/n : n \in \mathbb{N}\}$. (You may use any result in the book.)
#1 @ x is an upper bound for A \( \subseteq \mathbb{R} \) if
\[ x \geq a \ \forall a \in A \]

@ \( x \) is a LUB for A if \( x \leq y \ \forall \) upper bound \( y \) of A

@ Every set bded from above has a LUB

@ \( s_{n+1} \geq s_n \ \forall \ n \). Let \( s_0 = \text{LUB} \{ s_n \} \). We prove that \( s_n \to s_0 \). Fix \( \varepsilon > 0 \). We find \( N \) st \( n \geq N \implies |s_n - s_0| < \varepsilon \). But \( s_0 \), the \( \text{LUB} \{ s_n \} \)

implies \( s_0 \geq s_n \ \forall n \), and \( \exists N \) st \( s_N > s_0 - \varepsilon \).

Thus for \( n > N \) we have \( s_0 - \varepsilon < s_N \leq s_n \leq s_0 \),

so \( |s_n - s_0| < \varepsilon \) \( \checkmark \)
\[\sum_{n=1}^{s_n} S_n \rightarrow S_0 \text{ if } \forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n > N \mid S_n - S_0 \mid < \varepsilon\]

\[S_n \rightarrow +\infty \text{ if } \forall M > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n > N \quad S_n > M.\]

b) \[S_n \text{ is Cauchy if } \forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall m, n > N \quad |S_n - S_m| < \varepsilon\]

c) \[\neg (S_n \text{ is Cauchy}) \equiv \neg (\forall \varepsilon > 0 \exists N \in \mathbb{N} s.t. \forall m, n > N \mid S_n - S_m \mid < \varepsilon)\]

\[\equiv \exists \varepsilon > 0 \forall N \in \mathbb{N} \exists m, n > N \mid S_n - S_m \mid \geq \varepsilon\]
(2) Assume \( s_n \to s_0 \in \mathbb{R} \). We show \( s_n \) is Cauchy. Fix \( \varepsilon > 0 \). We find \( N \) s.t. \( m, n > N \Rightarrow |s_n - s_m| < \varepsilon \). Choose \( N \) s.t. \( n > N \Rightarrow |s_n - s_0| < \frac{\varepsilon}{2} \). Then \( m, n > N \Rightarrow |s_n - s_m| \leq |s_n - s_0 + s_0 - s_m| \leq |s_n - s_0| + |s_m - s_0| \)

\[ \varepsilon \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon \]

(2) Assume \( s_n \to +\infty \). Choose \( \varepsilon = 1 \).

We prove that \( \forall N \in \mathbb{N} \exists m, n > N \) s.t. \( |s_n - s_m| \geq \varepsilon \). Choose any \( n > N \). Then since \( s_n \to +\infty \), setting \( M = s_n + 1 \) we know \( \exists m > N \) s.t. \( s_m > M = s_n + 1 \). Thus \( |s_n - s_m| > 1 \), proving \( s_n \) not Cauchy.
\( S_N = \sup \{ S_n : n > N \} \)

\( \bar{S}_N = \inf \{ S_n : n > N \} \)

\( \bar{s} = \lim_{N \to \infty} \bar{S}_N \)

\( \underline{s} = \lim_{N \to \infty} S_N \)

\( \underline{s} \leq \bar{s} \leq \bar{S}_N \)

Prove: \( \underline{s} \leq S_N \)

\( \bar{S}_N = \sup \{ S_n : n > N \} \geq \sup \{ S_{M} : n > M \} = \bar{s}_M \)

if \( M \geq N \). \( \bar{S}_N \) is non-increasing sequ 

\( \Rightarrow \bar{S}_N \geq \underline{s} \) its limit.
\( S_n = (1 + (-1)^n) \cdot e^n \)

(a) \( \liminf S_n = 0 \), \( \limsup S_n = 2 \)

because \( e^n \to 1 \)

(b) \( S_{n_k} \to 2 \) for \( n_k = 2k \)

\#5 Example: Define \( S_n \) as follows —

\[
1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots
\]

\( S_1, S_2, S_3, S_4, \ldots \)

(other possible examples exist)

(b) \( A = \{ \frac{1}{n} : n \in \mathbb{N} \} \) cannot be \( S \) for any \( S_n \) because it is not closed — i.e.,

\( \frac{1}{n} \to 0 \) but \( 0 \notin A \).