Introduction to General Relativity
Math 280, UC-Davis - Temple

- <u>Grading</u>: Hand in FIP Homework problems
- <u>Office Hrs</u>: T-Th after class / 3148 MSB/Appt
- <u>Text</u>: <u>Introduction to General Relativity</u>
  by Adler, Bazin & Schiffer

Out of Print — copy of text together with brief article on history on file in Math Off.
No Textbook Required / Notes should be self-contained

- Articles & Notes can be found on my webpage: http:\www.math.ucdavis.edu

References:

(1) R. Wald, General Relativity (Standard)

(2) S. Weinberg, Gravitation & Cosmology: Principles and applications of GR (Great physics)

(3) Misner, Thorne & Wheeler, Gravitation ("Telephone book" - full of geometrical & physical insights)

(4) Hawking & Ellis, Large Scale Structure of Spacetime (Short on intuition / Ch 2 gives brief intro to diff. geom / Ch 6-10 covers singularity theorems)

(5) S. Chandrasekhar, The Mathematical Theory of Black Holes (Good advanced book)

(6) B. Dubrovin & A Fomenko Modern Geometry Methods & Applications Vol I (Great Exposition)

(7) Ø Grøn & S. Hervik <u>Einstein's General Theory of Relativity</u> (Great book/advanced)

(8) L. Hughston & K. Tod <u>An Introduction to General Relativity</u> (Excellent introduction)

(9) Sachs & Wu, <u>General Relativity for Mathematicians</u> (Avoid!)

(10) <u>Peebles</u>, Cosmology (standard)

Famous Classical Texts:

(1) Theory of Relativity, Eddington '20's

(2) Relativity, Thermodynamics and Cosmology
R. Tolman, Oxford Press, 1934

(3) Whittaker ≈ 1950 Credited Spec. Rel. to Lorentz & Poincaré.

Differential Geometry

(1) Differential Geometry, Vol I, II SPIVAK

(2) Foundations of Differentiable Manifolds & Lie Groups F. W. Warner

History

(1) Subtle is the Lord: the science and life of Albert Einstein, A. Paes

Special Relativity

(1) Spacetime Physics Taylor and Wheeler

Introduction:

☒ The Einstein Equation (1916):

(1) $$G = KT$$

- First and simplest covariant field equation for gravitational field
- Replaced Newton's Law of Gravitation (1687)

(2) $$\vec{F} = -G_0 \frac{M_1 M_2}{r^2} \vec{r}$$

- $$\underset{\substack{\nearrow \\ \text{Einstein} \\ \text{Curvature Tensor}}}{G} = K \underset{\substack{\nwarrow \\ \text{Stress Energy Tensor} \\ \text{(Einstein)}}}{T}$$

Here: $$K = \frac{8\pi G_0}{c^4},$$

is forced by the assumption that (1) and (2) agree in the limit of low velocities & weak gravitational fields: This is the

<u>Correspondence Principle</u>

History

☒ Fundamental Question: Why do the planets move around the sun?

Newton: Ans: planets go around the sun because the sun "pulls" on the planets with an inverse square "force" (1687)

$$\vec{F} = M\vec{a}$$

$$\vec{F} = -G_0 \frac{M_S M_P}{r^2} \hat{r}$$

Einstein: Ans: planets go around the sun because the enormous mass of the sun (99% of mass) causes spacetime to be curved and planets move in "st lines" thru a "curved space" (1916)

$$G = KT$$

Einstein Curvature Tensor

Stress Energy Tensor

"mass-energy & mass-energy flux causes curvature"

- In 1687 (Principia) Newton showed that these laws explained the 3 Kepler laws:

① Planets move in ellipses around sun with sun at one focus

② equal areas in equal time

③ $\frac{L^3}{T^2} = \text{const}$ indep of planet.

<u>Triumph</u> 1846 John Adams & Urbain Leverrier predict position of Neptune from deviations in orbit of Uranus from Newtonian prediction.

Problems with Newton Theory "Conceptually"

(1) "force" as "pull" across empty space?

(2) ∞ speed of propagation

(3) Mysterious "Equivalence Principle"

Grav. Mass = Inertial Mass

Ex: let $F = M_E \vec{a}$ = force exerted by sun on earth

$$\vec{F} = \cancel{M_E} \vec{a} = -G_0 \frac{M_S \cancel{M_E}}{r^2} \hat{r}$$

inertial mass that resists acceleration

gravitational mass appearing in 3rd law

Conclude: acceleration is independent of mass of earth: i.e., a feather would follow same trajectory as earth if given same initial conditions.

$\rightsquigarrow$ Planetary motions appear to be more a property of the space than of the object moving thru the space (eg, due to "force" on the object)

(4) $$\vec{F} = -G_0 \frac{M_s M_e}{r^2} \hat{r} = M_e \vec{a}$$

is a coordinate dependent expression.
For a system of particles with positions

$\underset{\sim}{x}_1, \dots, \underset{\sim}{x}_N \quad \underset{\sim}{x}_i = (x_i^1, x_i^2, x_i^3) = \underset{\sim}{x}_i(t)$

$$m_n \frac{d^2 \underset{\sim}{x}_n}{dt^2} = -G \sum_k \frac{m_n m_k (\underset{\sim}{x}_n - \underset{\sim}{x}_k)}{|\underset{\sim}{x}_n - \underset{\sim}{x}_k|^3}$$

Equations invariant under coord changes

$$\underset{\sim}{x}' = R\underset{\sim}{x} + \underset{\sim}{v}t + \underset{\sim}{d}$$
$$t' = t + \tau$$
$\Leftarrow$ Galileo Group

R arbitrary rotation (any orthogonal matrix $RR^t = \mathrm{Id}$)

But only these:

$\Leftrightarrow$ Galilean Invariance.

Theory is not coordinate indept: eg: what determines these frames, and why do the stars appear to (coincidentally) be fixed in these frames)

<u>Principle</u> <u>of</u> <u>Mach</u>: Somehow the masses in the universe determine the inertial frames

☒ <u>Einsteins</u> <u>Theory</u> resolves these problems!

- Assumption: all properties of gravitational field are determined by a sg(−1,1,1,1) metric g on spacetime: in any coord system, $\underset{\sim}{x}$, g is a matrix field

$$g_{ij}(\underset{\sim}{x}) \equiv 4\times 4 \text{ symmetric, nonsingular, bilinear form of } \operatorname{sgn}(-1,1,1,1)$$

- freefall paths: geodesics of g
- "aging time" for travellor along path $\underset{\sim}{x}(s)$ ≡ arclength of path as determined by g
- Non-rotating coordinate frames determined by ∥-translation given by connection assoc. with g.

Q: What is the equation that the gravitational field g_{ij} should satisfy?

Einstein: $G = KT$

G: Einstein Curvature Tensor

T: Stress Energy Tensor

• Fundamental Result: 1859 Riemann defined the <u>curvature</u> ~~in the~~ associated with a Riemannian metric g at each point.

$\Rightarrow$ Riemann Curvature Tensor.

- In a coordinate system $\underset{\sim}{x} = (x^0, x^1, x^2, x^3)$ on spacetime, $x^0 = ct$

$$G_{ij} = \kappa T_{ij} \qquad i\,j = 0, \dots 3$$

$$G_{ij} = R_{ij} - \frac{1}{2} R\, g_{ij}$$

G_{ij} and T_{ij} are 4x4 matrix (or tensor) fields defined on spacetime

R_{ij} ≡ Ricci Curvature tensor

R ≡ scalar curvature

g_{ij} ≡ spacetime metric

inverse of g_{ij}

$$R_{ij} = R^{\sigma}_{i\sigma j}, \quad R = R^{\sigma}_{\sigma} = g^{\sigma\tau} R_{\tau\sigma}$$

Einstein Summation Convention: sum repeated up-down indices from 0 to 3

R^i_{jkl} ≡ Riemann Curvature Tensor (1851)

$$R^i_{jkl} = \underbrace{\Gamma^i_{jl,k} - \Gamma^i_{jk,l}}_{\text{"curl"}} + \underbrace{\Gamma^i_{\sigma k}\Gamma^{\sigma}_{jl} - \Gamma^i_{\sigma l}\Gamma^{\sigma}_{jk}}_{\text{"bracket"}}$$

$$,k \equiv \frac{\partial}{\partial x^k}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma} \{ -g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j} \}$$

Conclude: G_{ij} is a complicated function of the metric entries g_{ij} and their derivatives up to order 2.

- In empty space, $T_{ij} = 0 \Rightarrow$

Einstein: $G_{ij}[g] = 0$ $\left\{ \begin{array}{l} \text{2nd order PDE} \\ \text{in } g_{ij} \end{array} \right.$

Newton: $\Delta \Phi = 0$ $\left\{ \begin{array}{l} \text{2nd order PDE} \\ \text{in grav. potential} \end{array} \right.$

Φ = gravitational potential:

$\nabla \Phi = - \frac{\text{force}}{\text{mass}}$ at $p \in \mathbb{R}^3$

- For a perfect fluid,

$$T_{ij} = (\rho c^2 + p)\, u_i\, u_j + p\, g_{ij}$$

$\rho \equiv \dfrac{\text{mass-energy}}{\text{3-vol}}$ in "frame of particle"

$u^i \equiv \dfrac{dx^i}{d\tau} \equiv$ "4-velocity of fluid particle"

$p \equiv$ pressure

Einstein : $\quad G_{ij} = K\, T_{ij}$ $\quad \left\{ \begin{array}{l} \text{source involves} \\ \text{mass and momentum} \\ \text{densities and their fluxes} \end{array} \right.$

$G_0 = 1$

Newton : $\quad \Delta \Phi = 4\pi \rho$ $\quad \left\{ \begin{array}{l} \rho \equiv \text{mass density} \\ \text{at fixed time.} \end{array} \right.$

- In limit of low velocities, the (0,0)-eqn

$$G_{00} = K\, T_{00} \longrightarrow \Delta \Phi = 4\pi \rho \quad \checkmark$$

- Problem with Newton – RHS $\rho(x,t)$ must evolve according to some equation which expresses cons. of mass & momentum – but Φ is determined instantaneously, indept of derivatives of ρ.

- Indeed: In the classical case,

cons. of mass: $\boxed{\rho_t + \operatorname{div} \rho u = 0}$ 1 equn

$u = \frac{d\underset{\sim}{x}}{dt}$ = 3-velocity of fluid particle.

Newton's law: ($\Omega(t)$ moving with fluid)

$$\frac{d}{dt}\int_{\Omega(t)} \rho u^i \, d^3\underset{\sim}{x} = -\int_{\partial\Omega} p n^i \, d^2S - \int_{\Omega(t)} \nabla\Phi \rho \, d^3\underset{\sim}{x}$$

n^i: outer normal

(grav. force on $\rho d^3\underset{\sim}{x}$ is $\nabla\Phi \rho d^3\underset{\sim}{x}$)

$$\Leftrightarrow \int_{\Omega(t)} (\rho u^i)_t + \operatorname{div}(\rho u^i) \, d^3\underset{\sim}{x} = \int_{\Omega(t)} -\frac{\partial}{\partial x^i} p - \nabla\Phi \rho \, d^3\underset{\sim}{x}$$

$$\Leftrightarrow \boxed{(\rho u^i)_t + \operatorname{div}(\rho u^i + p e^i) = -\rho \nabla\Phi^i} \quad \text{3 equn's}$$

$\overset{\text{Newton}}{\Rightarrow}$ 4 equn's in 4 unknowns ρ, u^i, Φ.

- For Einstein, the conservation equations are built into $G_{ij} = \kappa T_{ij}$ automatically because G_{ij} is constructed to satisfy

$$\mathrm{div}\, G = 0$$

$$\Rightarrow \qquad \mathrm{div}\, T = 0$$

$\Leftrightarrow$ the relativistic Euler equations which express cons. of mass & energy.

$$T^{ij} = (\rho c^2 + p) u^i u^j + p g^{ij}$$

$$G = G[g^{ij}]$$

$g^{ij} \equiv$ gravitational metric tensor

Possible Topics:

I. Differential Geometry

- M^4 Manifold of events
- Tangent bundle / Cotangent bundle / Tensors / Differential Forms / Lie Derivatives
- Integration / maps / pullback / orientation
- Metric / ‖-translation / Connection / Covariant Derivatives / Curvature / Hodge-$*$

II Einsteins Theory

- Special Relativity
- Stress Energy Tensor / Perfect Fluid / E&M
- $G = kT$ / Newtonian Limit / Red Shift / Fermi Transport / Coriolis force
- Special Solutions / Schwarzshild Soln / Birkhoff Thm / Max' Symm Spaces / Robertson-Walker Metric / Interior Schwarz. / DiSitter spaces / Oppenheimer Snyder
- Stability

- Initial value problem / well-posedness / equation count - guage freedom / Bianci identities
- Shock-Waves / Special Relativity / Te-Smo Result