Relativist & non-relativistic fluids

- Compressible Euler Equations $\approx$ Newton's Laws for a Continuous Media

Non-relativistic = classical version

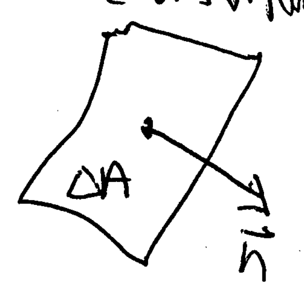
(MA) $\rho_t + \operatorname{div} \rho\vec{v} = 0$ [cons of mass = continuity eqn]

(MO^i) $(\rho v^i)_t + \operatorname{div}[(\rho v^i)\vec{v} + p e^i] = 0$

Here: $\rho = \frac{\text{mass}}{\text{vol}}$, $\vec{v} = (v^1, v^2, v^3)$ = velocity

2-d surface

$p = \frac{\text{force}}{\text{area}}$ exerted by fluid



force $\approx p \cdot \Delta A\, \vec{n}$

- Look to solve for $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$ starting from i-conds, ∂ conds, etc.
  - Unknowns: v^1, v^2, v^3, ρ, p

  Equation: 4

Need Eqn of state to close — barotropic $p = p(\rho)$

- The divergence thm tells us what they mean:

- General case: $q = \frac{\text{stuff}}{\text{vol}}$ is transported at velocity $\vec{v} = (v^1, v^2, v^3)$

Defn: $q\vec{v} \equiv$ "stuff flux vector"

Claim: $$q_t + \operatorname{div}(q\vec{v}) = 0 \qquad (*)$$

expresses "conservation of q"

I.e. Integrate (*) over 3-vol V at fixed t:

$$\iiint_V q_t + \operatorname{div}(q\vec{v})\, \underbrace{dV}_{dx^1 dx^2 dx^3} = 0$$

$$\frac{d}{dt}\iiint_V q\, dV + \iiint_V \operatorname{div}(q\vec{v})\, dV = 0$$

$$\Leftrightarrow \quad \underbrace{\frac{d}{dt}\iiint_{V} q\, dV}_{\substack{\text{mass} \\ \text{in } V}} + \underbrace{\iint_{S} q\, \underbrace{\vec{v}\cdot\vec{n}}_{\frac{\text{distance}}{\text{time}} \text{ normal to } S}\, dS}_{\substack{\text{rate at which stuff} \\ \text{passes outward thru} \\ \text{Boundary of } V \,\equiv\, S}} = 0$$

<u>Conclude</u>: $q_t + \operatorname{div} q\vec{v} = 0$ expresses that the time rate of change of stuff in any volume V = rate at which stuff passes outward thru ∂V. $\Leftrightarrow$

<u>Conservation</u> <u>of</u> <u>Stuff</u>

• Conclude:

(MA) $$\rho_t + \operatorname{div} \rho \underset{\sim}{v} = 0 \equiv \frac{d}{dt}\iiint_{\mathcal{V}} \rho \, dV = -\iint_{\partial\mathcal{V} \equiv \mathcal{S}} \rho \vec{v}\cdot\vec{n}\, dA$$

$\equiv$ conservation of mass

$\equiv$ continuity equn.

(mo^i) $$(\rho v^i)_t + \operatorname{div}\big(\underbrace{\rho v^i \vec{v}} + p \underset{\sim}{e}^i\big) = 0$$

$\rho u^i \equiv \dfrac{\text{mass} \times \text{vel}}{\text{vol}}$ $\qquad (\rho v^i)\vec{v} \equiv$ i-mom flux vector

$\equiv$ i-mom density.

Use Div Thm to Interpret:

outward normal

$$\underbrace{\frac{d}{dt}\iiint_{\mathcal{V}} \rho u^i \, dv}_{\text{i-mom in vol } \mathcal{V}} = -\underbrace{\iint_{\mathcal{S}} \rho u^i u\cdot\vec{n}\, ds}_{\text{rate at which i-mom passes outward thru } \mathcal{S} = \text{boundary of } \mathcal{V}} - \underbrace{\iint_{\mathcal{S}} (p e^i\cdot\vec{n})\, ds}_{\text{i-comp of the force on } \mathcal{S} \times (-1)}$$

- <u>Conclude</u>: (CE) $\Leftrightarrow$ Cons of Mass (MA)

& (MO) $\equiv$ changes in momentum are due only to the pressure forces

<u>The equations</u>:

$$\begin{pmatrix} \rho \\ \rho v^1 \\ \rho v^2 \\ \rho v^3 \end{pmatrix}_t + \operatorname{div} \begin{pmatrix} \rho v^1 & \rho v^2 & \rho v^3 \\ \rho v^1 v^1 + p & \rho v^1 v^2 & \rho v^1 v^3 \\ \rho v^2 v^1 & \rho v^2 v^2 + p, & \rho v^2 v^3 \\ \rho v^3 v^1 & \rho v^3 v^2 + & \rho v^3 v^3 + p \end{pmatrix}$$

or

$$\operatorname{Div}_{t,\vec{x}} \underbrace{\begin{pmatrix} \rho & \rho v^1 & \rho v^2 & \rho v^3 \\ \rho v^1 & \rho v^1 v^1 + p & \rho v^1 v^2 & \rho v^1 v^3 \\ \rho v^2 & \rho v^2 v^1 & \rho v^2 v^2 + p & \rho v^2 v^3 \\ \rho v^3 & \rho v^3 v^1 & \rho v^3 v^2 & \rho v^3 v^3 + p \end{pmatrix}^{\alpha\beta}}_{4\times 4 \text{ symmetric Tensor } T^{\alpha\beta} = T_{CE}^{\alpha\beta}}$$

- For GR, Einstein wants eqvn of form

$$G = \kappa T$$

Looks for $\mathrm{Div}\, G = 0$ so

$$\mathrm{Div}\, T = 0$$

is the relativistic version of compressible Euler

<u>Claim</u>: The relativistic version of T_{CE} is

$$T^{\alpha\beta} = (\rho + P)\, u^{\alpha} u^{\beta} + P g^{\alpha\beta}$$

$\uparrow$

metric defines the gravitational field.

No Gravity = Flat space $\quad g^{\alpha\beta} = \eta^{\alpha\beta} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$

- Our goal: "Prove" that as the speed of light $c \to \infty$

$$\mathrm{Div}\, T^{\alpha\beta} = 0$$ relativistic compressible Euler equations

tend to

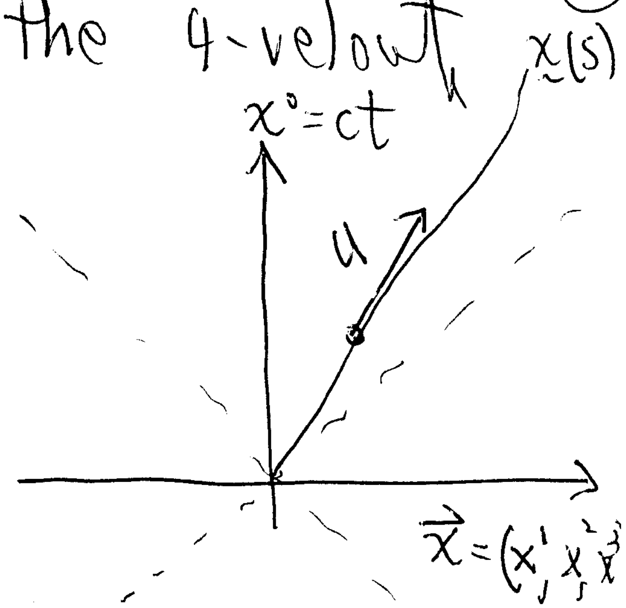
$$\mathrm{Div}\, T_{CE}^{\alpha\beta} = 0$$ classical compressible Euler equations.

- To make the connection:

Relativistically $\rho = \frac{\text{energy}}{\text{vol}} \approx \rho c^2$ (ρ: $\frac{\text{mass}}{\text{vol}}$)

Thus if $\rho \equiv \frac{\text{mass}}{\text{vol}}$ of CE, we assume the energy is mostly rest energy $\Rightarrow$ replace $\rho \to \rho c^2$

• $u = (u^0, u^1, u^2, u^3)$ is the 4-velocity of the particles



$\underset{\sim}{x}(s)$ = particle path

$u = \dfrac{d\underset{\sim}{x}}{ds}$ = 4-velocity

$\underset{\sim}{x} = (x^0, x^1, x^2, x^3)$

Here: ds is arclength wrt Minkowski Metric $\eta = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$.

Assume: $\boxed{u^0 = \sqrt{1 + \left(\frac{v}{c}\right)^2}} = 1 + O\left(\frac{v}{c}\right)^2$

Define: $\boxed{u^i = \dfrac{v^i}{c}}$ v^i = classical velocity

To justify —

$$ds^2 = -d(x^0)^2 + d(x^1)^2 + d(x^2)^2 + d(x^3)^2$$

Minkowski metric

Now $x^0 = ct$, t = classical time

$s = c\tau$, τ = propertime

Divide by ds^2:

$$1 = -\left(\frac{dx^0}{ds}\right)^2 + \left(\frac{dx^1}{ds}\right)^2 + \left(\frac{dx^2}{ds}\right)^2 + \left(\frac{dx^3}{ds}\right)^2$$

put − sign here for timelike vectors

$$\Rightarrow \left(\frac{dx^0}{ds}\right)^2 = 1 + \sum_{i=1}^{3}\left(\frac{dx^i}{ds}\right)^2$$

u

$$\frac{dx^0}{ds} = u^0 = \sqrt{1 + \frac{v^2}{c^2}} = \left(1 + O\left(\frac{v^2}{c^2}\right)\right)$$

$$\frac{dx^i}{ds} = \frac{dx^i}{c\,d\tau} \approx \frac{v^i}{c}$$

Taylor

• Thus compute $\mathrm{Div}\, T^{\alpha\beta} = T^{\alpha\beta}{}_{,\beta} = \left[(\rho+p)u^\alpha u^\beta + p\eta^{\alpha\beta}\right]_{,\beta}$
with:

$$\rho \to \rho c^2 \qquad \text{"}\frac{\text{mass}}{\text{vol}}\text{"}$$

$$u^0 \to \sqrt{1+\cancel{\beta}\left(\frac{v}{c}\right)^2} = 1 + O\left(\frac{v^2}{c^2}\right)$$

$$u^i \to \frac{v^i}{c}$$

$$T^{0\beta} = \left((\rho c^2+p)u^0u^0 + p\eta^{00},\ \rho c^2 u^0 u^i\right) \qquad \beta = (0,i)$$

$$= \left((\rho c^2+p)\left(1+\left(\frac{v}{c}\right)^2\right) - p,\ \rho c^2 \frac{\vec{v}}{c}\sqrt{1+\left(\frac{v}{c}\right)^2}\right)$$

$$= \left(\rho c^2\left(1+\left(\frac{v}{c}\right)^2\right),\ \rho c \vec{v}\sqrt{1+\left(\frac{v}{c}\right)^2}\right)$$

Assume derivatives of v are order v, & v, p, ρ are $O(1)$ as $c \to \infty$

$$\mathrm{Div}\,T^{0\beta} = T^{0\beta}{}_{,\beta} = \left(\rho c^2\left(1+\left(\frac{v}{c}\right)^2\right)\right)_{,x^0} + \mathrm{div}\left(\rho c \vec{v}\sqrt{1+\left(\frac{v}{c}\right)^2}\right) \qquad ct = x^0$$

$$= (\rho c^2)_{,x^0} + \mathrm{div}(\rho c\vec{v}) + O\left(\frac{v}{c}\right) = \ldots$$

$$\cdots = c\left\{\rho_t + \operatorname{div}\rho\vec{v}\right\} + O\left(\frac{v^2}{c}\right) = 0$$

$$\Leftrightarrow \quad \rho_t + \operatorname{div}\rho\vec{v} + O\left(\frac{v^2}{c}\right) = 0$$

Take $c \to \infty$ & get $\boxed{\rho_t + \operatorname{div}\rho\vec{v} = 0}$ (MA)

- $T^{iB} \underset{\substack{\uparrow \\ i=1,2,3}}{=} (\rho+p)u^i u^B + p\,\eta^{iB}$ $\qquad \rho \to \rho c^2$, $\quad u^0 \to \sqrt{1+\left(\frac{v}{c}\right)^2}$

$$= \Big(\underbrace{\rho c^2\sqrt{1+\left(\tfrac{v}{c}\right)^2}\,\frac{v^i}{c}}_{B=0},\ \underbrace{\frac{\rho \cancel{c^2} v^i\vec{v}}{\cancel{c^2}} + p e^i}_{B=1,2,3}\Big)$$

$$\operatorname{Div} T^{iB} = T^{iB}{}_{,B} = T^{i0}{}_{,0} + T^{ij}{}_{,j} \qquad j=1,2,3$$

$$= \left\{\rho c^2\sqrt{1+\left(\frac{v}{c}\right)^2}\,\frac{v^i}{c}\right\}_{\substack{\\ ct = x^0}} + \operatorname{div}\left\{\rho \cancel{c^2}\frac{v^i\vec{v}}{\cancel{c^2}} + p e^i\right\}$$

$$= \left\{\rho v^i\sqrt{1+\left(\frac{v}{c}\right)^2}\right\}_t + \operatorname{div}\left\{\rho v^i\vec{v} + p e^i\right\} = 0$$

Conclude:

$$\mathrm{Div}\, T^{i\beta}=0 \Leftrightarrow (\rho v^i)_t + \mathrm{div}\{\rho v^i \vec{v} + p e^i\} + O\left(\frac{v}{c}\right)^2 = 0$$

In limit $c\to\infty \Rightarrow$ (MO^i)

While this is correct it is incomplete. The classical limit of ~~Einstein~~ Rel Euler is more subtle.

Problem: Barotropic fluids are of fundamental importance in GR but are too simple for many applications. They provide an excellent model problem, but ~~they~~ $p=p(\rho)$ doesn't allow for temperature dependence. <Important Barotropic fluid is Pure Radiation: $\boxed{p=\frac{1}{3}\rho}$ Barotropic>

- Taking the classical limit $c \to \infty$ for general perfect fluids (Only pressure acts)

- Start with classical Euler:

(MA) $\rho_t + \operatorname{div}(\rho \vec{v}) = 0$

(Mo^i) $(\rho v^i)_t + \operatorname{div}(\rho v^i \vec{v} + p e^i) = 0$

What about conservation of energy?

(En) $E_t + \operatorname{div}[(E+P)\vec{v}] = 0$

Here: $E = \dfrac{\text{total energy}}{\text{vol}}$

Assumption: $E = \underbrace{\frac{1}{2}\rho v^2}_{\substack{\text{kinetic} \\ \text{energy of motion} \\ \text{per vol}}} + \rho e$

$e = \begin{cases} \text{specific} \\ \text{internal} \\ \text{energy} \end{cases}$

$v = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}$

<u>Language</u>: "specific" always means "per mass" while "density" is "per volume"

<u>Thus</u>: $\rho e = \frac{\text{mass}}{\text{vol}} \cdot \frac{\text{energy}}{\text{mass}} = \frac{\text{energy}}{\text{vol}}$

So ρe is the internal energy density.

For a perfect fluid, the temperature T is some given function of e alone–

$e = c_v T$ for an ideal gas.

Divergence Thm :

$$\iiint_{\mathcal{V}} E_t + \operatorname{div}(E\vec{v} + p\vec{v})\, dV = 0$$

$$\underbrace{\frac{d}{dt}\iiint_{\mathcal{V}} E\, dV}_{\substack{\text{time rate of} \\ \text{chng of } E \text{ in } \mathcal{V}}} \;\cancel{=}\!- \underbrace{\iint_{\cancel{S}=\partial\mathcal{V}} E\vec{v}\cdot\vec{n}\, dS}_{\substack{\text{flux of } E \\ \text{thru } \partial\mathcal{V}=S}} - \underbrace{\iint_{\cancel{S}=\mathcal{V}} p\vec{v}\cdot\vec{n}\, dS}_{\substack{\frac{\text{Work}}{\text{Time}} \text{ done by} \\ \text{pressure on } \partial\mathcal{V}}}$$

$$[p\vec{v}\cdot\vec{n}] = \frac{\text{Force} \times \text{Dist}}{\text{Area Time}}$$

- The relativistic compressible Euler eqn's —

Introduce the number density:

Freisteuhler/Temple
Sept, 2014

$$n = \frac{\#\text{ particles}}{\text{vol}}$$

Assume each particle has mass m so

$$mn = \frac{\text{mass}}{\text{density}}$$

The rest energy of each particle (by $E = mc^2$) is

$$mnc^2 = \frac{\text{energy of rest mass}}{\text{vol}}$$

But moving particles have extra kinetic energy

$$\rho = \frac{\text{energy}}{\text{vol}} = \underbrace{mnc^2}_{\substack{\text{rest}\\ \text{energy}}} + \underbrace{ne}_{\substack{\text{internal}\\ \text{energy} \approx \text{KE}}}$$

(Extreme relativistic limit $\rho \approx ne$
classical limit $\rho \approx mnc^2$)

Relativistic Euler with # density —

$$\mathrm{Div}\, T^{\alpha\beta} = T^{\alpha\beta}{}_{;\beta} = 0$$

$$T = (\rho + p) u^i u^j + p g^{ij} \quad \text{(as before)}$$

$$\mathrm{Div}(nu) = (nu)^{\alpha}{}_{;\alpha} = 0$$

Conservation of particle number.

Here: $q \equiv \frac{\text{stuff}}{\text{vol}}$ $\quad u = $ 4-velocity

qu = "stuff flux vector"

$\mathrm{Div}(qu) = 0$ relativistic version of cons of "stuff".

To take the classical limit:

$$\rho = \underbrace{mnc^2}_{\equiv \rho_{CE}} + ne$$

& view new mn as classical mass density

$$u^0 = \sqrt{1 + \left(\frac{v}{c}\right)^2} = 1 + O\left(\frac{v}{c}\right)^2$$

$$u^i = \frac{v^i}{c}$$

Plug into equations & get:

$$T^{\alpha\beta}{}_{,\beta} = \frac{1}{c}\left\{(nmc^2 + ne + p)u^\alpha u^0 + p\eta^{\alpha 0}\right\}_{,t}$$

$$+ \left\{(\underbrace{nmc^2 + ne}_{\rho} + p)u^\alpha \frac{v^j}{c} + p\eta^{\alpha j}\right\}_{,x^j}$$

Taking $\alpha = 0$ obtain:

$$T^{0B}{}_{,B} = \frac{1}{c}\left[\left\{(nmc^2+ne+p)\left(1+\left(\frac{v}{c}\right)^2\right)-p\right\}_{\cancel{c}t}\right.$$

$$\left. +\left\{(nmc^2+ne+p)\sqrt{1+\left(\frac{v}{c}\right)^2}\,\frac{v^j}{\cancel{c}}\right\}_{x^j}\right]$$

$$= \frac{1}{c}\left[c^2\left\{(nm)_t + (nmv^j)_{x^j}\right\}\right.$$

$$\left. + \left\{\underbrace{(nmv^2+ne)_t}_{\text{Looks like E but missing fact of } \frac{1}{2}!} + \left(\left(\tfrac{1}{2}mv^2+ne+p\right)v^j\right)_{x^j}\right\} + O\left(\frac{v}{c}\right)^2\right]$$

Taking $\alpha = i = 1,2,3$

$$T^{iB}{}_{,B} = \frac{1}{c}\left((nmc^2+ne+p)\sqrt{1+\left(\frac{v}{c}\right)^2}\,\frac{v^i}{c}\right)_t$$

$$+\left((nmc^2+ne+p)\frac{v^i v^j}{c^2} + p\delta^{ij}\right)_{x^j}$$

$$= \left\{(nmv^i)_t + (nmv^iv^j + p\delta^{ij})_{x^j}\right\} + O\left(\frac{v}{c}\right)^2$$

For the number density -

$$(nu^B)_{,B} = \frac{1}{c}\left(n\sqrt{1+\left(\frac{v}{c}\right)^2}\right)_t + \left(n\frac{v^j}{c}\right)_{x^j}$$

$$= \frac{1}{c}\left\{n_t + (nv^j)_{x^j}\right\} + \frac{1}{c^3}\left(\frac{1}{2}nv^2\right)_t + O\left(\frac{1}{c}\left(\frac{v}{c}\right)^4\right)$$

$$\sqrt{1+\left(\frac{v}{c}\right)^2} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$$

Collecting we obtain:

$$0 = T^{0B}{}_{,B} = c\left\{(nm)_t + (nmv^j)_{x^j}\right\} \qquad (1)$$

$$+ \frac{1}{c}\left\{(\underbrace{nmv^2}_{\text{need } \frac{1}{2}mv^2} + ne)_t + (\underbrace{(\tfrac{1}{2}mv^2 + ne + p)v^j}_{(E+P)\vec{v}})_{x^j}\right\} + O\left(\frac{v^4}{c^3}\right)$$

$$0 = T^{iB}{}_{,B} = (nmv^i)_t + (nmv^iv^j + p\delta^{ij})_{x^j} + O\left(\frac{v}{c}\right)^2 \quad (2)$$

$$0 = (nu^B)_{,B} = \frac{1}{c}\left\{n_t + (nv^j)_{x^j}\right\} + \frac{1}{c^3}\left(\frac{1}{2}nv^2\right)_t + O\left(\frac{1}{c}\left(\frac{v}{c}\right)^4\right)$$

(3)

- To obtain CE equations –
- Mult (3) by mc to get

$$(mn)_t + (nmv^j)_{x^j} + \frac{1}{c^2}\left(\frac{1}{2}mnv^2\right)_t = O\left(\frac{v}{c}\right)^4 \qquad (4)$$

taking $\rho \equiv mn = \frac{mass}{vol}$ in CE, and sending $c \to \infty$ gives (MA)

(eg now $\rho \equiv \rho_{CE}$ is classical mass density)

$$\rho_t + \mathrm{div}(\rho\vec{v}) = 0$$

- Setting $\rho \equiv mn$ in (2) gives $(MO)^i$

$$(\rho v^i)_t + \mathrm{div}(\rho v^i \vec{v} + p e^i) = 0 \quad \checkmark$$

- <u>Most interesting part</u> use (4) to substitute

$$(mn)_t + (nmv^j)_{x^j} = -\frac{1}{c^2}\left(\frac{1}{2}mnv^2\right)_t + O\left(\frac{v}{c}\right)^4 \qquad (5)$$

into (1) as follows:

22

Mult (5) by c^2 to get:

$$c^2\left\{(mn)_t + (mnv^j)_{x^j}\right\} + \frac{1}{2}mnv^2 = O\left(\frac{v}{c}\right)^2 \qquad (6)$$

Mult (1) by c to get:

$$c^2\left\{(mn)_t + (mnv^j)_{x^j}\right\}$$

$$+\left\{(\underbrace{nmv^2}_{\text{off by factor } \frac{1}{2}} + ne)_t + ((\underbrace{\tfrac{1}{2}mv^2 + ne}_{E} + p)v^j)_{x^j}\right\}$$

$$+ O\left(\frac{v}{c}\right)^2 = 0 \qquad (7)$$

Subtract (6) from (7)

$$\{(\overset{\text{Sd}}{\widetilde{nmv^2}} + ne)_t + ((\tfrac{1}{2}mv^2 + ne + p)v^j)_{x^j} = O\left(\frac{v}{c}\right)^2$$

$-\frac{1}{2}nmv^2$ from (6) !

$$c \to \infty \iff \boxed{E_t + \operatorname{div}((E+p)\vec{v}) = 0} \quad (En) \checkmark$$

- <u>Conclude</u>: It is pretty astounding how cons. of energy in the Newtonian sense is recovered from Einstein's relativistic theory $Div\left((\rho+p)u^\alpha u^\beta + p\eta^{\alpha\beta}\right)=0$

Indeed: a term from the number density equation that shows up due to the relativistic correction

$$u^0 = \sqrt{1+\left(\frac{v}{c}\right)^2} = 1 + \underbrace{\frac{1}{2}\left(\frac{v}{c}\right)^2}_{} + hot$$

in the number density eqn, is just the deficit needed in the energy eqn to get $E_t + div\left((E+p)\vec{v}\right) = 0$ <u>after</u> the leading order cons of number cancels !!!

☐ To close the relativistic & non-rel compress. Euler equations, we need equation of state:

Consider non-rel:

(MA) $\rho_t + \mathrm{div}(\rho \vec{v}) = 0$

(Mo^i) $(\rho v^i)_t + \mathrm{div}(\rho v^i \vec{v} + p e^i) = 0$

(En) $E_t + \mathrm{div}[(E+p)\vec{v}] = 0 \qquad E = \frac{1}{2}\rho v^2 + \rho e$

<u>Unknowns</u>: $\underbrace{\rho, p, e}_{\text{thermodynamic variables}}, v^1, v^2, v^3 = 6$

<u>Equations</u>: $= 6$

Principle of Thermo: Any two indept thermo variables should determine all others

$\rho, p, e, T, s \qquad s \equiv$ specific entropy

(a) <u>For example</u>:

Ideal Gas: $p\bar{V} = n\overset{\text{gas const}}{R}T$ $\bar{V}$ ≡ volume in equilibrium

n = # of particles

$\rho = \frac{n}{\bar{V}} \Rightarrow pv = RT$

$v = \frac{1}{\rho}$ ≡ specific vol.

(not vel anymore!)

<u>Eg</u>: simplest way to close —

$$\boxed{\begin{aligned} p &= RT\rho = p(T,\rho) \\ e &= C_v T \quad C_v = \frac{R}{\gamma - 1} \end{aligned}}$$

$\gamma - 1 = \frac{2}{3r}$ for a molecule consisting of r atoms

determines everything in terms of T, ρ.

- Usually the state variable $s = \frac{\text{entropy}}{\text{mass}}$ is used $\equiv$ specific entropy.

To get this assume 2nd Law of Thermo:

$$de = T\,ds - p\,dv$$

Means : $e = e(s,v)$ with $\frac{\partial e}{\partial s} = T$, $\frac{\partial e}{\partial v} = -p$

or : $ds = \frac{1}{T}de + \frac{p}{T}dv$

$s = s(e,v)$ with $\frac{\partial s}{\partial e} = \frac{1}{T}$, $\frac{\partial s}{\partial v} = \frac{p}{T}$

Theorem: Assume the Ideal Gas Law

$$pv = RT \qquad v = \frac{1}{\rho} \tag{1}$$

and assume

$$e = c_v T \qquad c_v = \frac{R}{\gamma - 1} \tag{2}$$

and 2nd Law:

$$de = T\,ds - p\,dv \tag{3}$$

Then:

$$e = \frac{c_v}{v^{\gamma-1}}\, e^{s/c_v} = e(s, v) \tag{4}$$

$$p = -\frac{\partial e}{\partial v} = c_v(\gamma - 1)\,\frac{1}{v^{\gamma}}\,\exp(s/c_v) \tag{5}$$

Proof: (1)-(5) is the equation of state for a polytropic = γ-law gas. Air at STP $\gamma \approx \cancel{5/3}$
1.4

Pf. (5) follows from (4) by (3). We need to integrate (3), which is a diff eqn for e

$$\frac{\partial e}{\partial s}(s,v) = T, \quad \frac{\partial e}{\partial v}(s,v) = -p$$

- Integrate by use of <u>free</u> <u>energy</u> ψ:

$$\psi = e - sT$$

$$d\psi = de - s\,dT - \underbrace{T\,ds}_{de + p\,dv}$$

$$= \cancel{de} - s\,dT - \cancel{de} - p\,dv$$

$$\boxed{d\psi = -s\,dT - p\,dv}$$

$$\frac{\partial \psi}{\partial T}(T,v) = -s, \quad \frac{\partial \psi}{\partial v}(T,v) = -p$$

The point: $p = \frac{RT}{V}$ ⇒ can int.

$$\frac{\partial\psi}{\partial v}(T,v) = -p = -\frac{RT}{V}$$

$$\boxed{\psi(T,v) = -RT\ln v + g(T)}$$

for some fn $g(T)$. But

$$s = -\frac{\partial\psi}{\partial T} = -\left(-R\ln v + g'(T)\right)$$

$$C_vT = e = \psi + sT = -\cancel{RT\ln v} + g(T) + \cancel{RT\ln v} - Tg'(T)$$

$$C_vT = g(T) - Tg'(T)$$

$$C_v = \cancel{g'(T)} - \cancel{g'(T)} - Tg''(T)$$

$$g''(T) = -\frac{C_v}{T} \Rightarrow g'(T) = -C_v\ln T + \text{const}$$

Thus: $S = R\ln v - g'(T)$

$= R\ln v + C_v \ln T + \cancel{\text{const}}$ only chnges in entropy matter

$= C_v \{(\gamma-1)\ln v + \ln T\}$

$S = C_v \ln(v^{\gamma-1} T) \iff T = v^{1-\gamma} e^{S/C_v}$

But $e = C_v T$ so Conclude:

$$e = C_v T = C_v \frac{1}{v^{\gamma-1}} e^{S/C_v} = e(S,v)$$

as claimed $\square$

<u>Note</u>: by (1)-(5) you can solve for all thermo var's in terms of any two

<u>Note</u>: Air = N_2 & O_2 $\Rightarrow r = 2 \Rightarrow \gamma - 1 = \frac{2}{3r} = \frac{1}{3}$
$\Rightarrow \gamma = \frac{4}{3}$. But γ 1.4 ??

Note: for a diffuse "non-interacting" gas (like air) you can count the # of degrees of vibrational freedom "d" that can store internal energy e. Assuming "equipartition of energy" & T = "the energy stored in the vibrations" ⇒

$$e = C_v T \;,\; C_v = \frac{R}{\gamma - 1} \;,\; \gamma - 1 = \frac{2}{d}$$

For a gas of r molecules, d = 3r = "rotational & translational degrees of freedom for vibrations".

But: QM ⇒ at low temp's, energy levels freeze out one degree. Thus —

$$N_2 + O_2 \Rightarrow \gamma - 1 = \frac{2}{3 \cdot 2 - 1} = 1.4$$

One of the descrepencies that led to Q Mech's

② Relativistic Fluids —

$$\rho = mnc^2 + ne \qquad (6)$$

$$de = T\,ds - P\,dv \qquad v = \frac{1}{mn} \qquad (7)$$

Similar Theory treating $\rho_{cl} = mn$, $v = \frac{1}{\rho_{cl}} = \frac{1}{mn}$

Then:

$$Pv = RT, \quad e = C_v T, \quad de = T\,ds - P\,dv \qquad (8)$$

lead to same conclusion (4), (5).

② The most relativistic fluid of all:

Pure Radiation (No classical analogue)

Assume: All energy is radiation, no rest mass energy $\Rightarrow$

$$\boxed{\underbrace{\mathrm{div}\left[(\rho+p)u^\alpha u^\beta + p\,\eta^{\alpha\beta}\right]}_{=0}} \quad \boxed{\rho = ne} \quad \boxed{p = \frac{1}{3}\rho} \qquad (9)$$