Classical Limit of Relativistic Euler —

Classical Euler:

(MA) $\rho_t + \mathrm{div}(\rho \vec{v}) = 0$

(Mo) $(\rho v^i)_t + \mathrm{div}(\rho v^i \vec{v} + p\vec{e}^{\,i}) = 0$

(En) $E_t + \mathrm{div}((E+p)\vec{v}) = 0 \qquad E = \frac{1}{2}m\vec{v}^{\,2} + \rho e$

$\uparrow \frac{\text{energy}}{\text{vol}}$

Rel Euler: $g_{ij} = \eta_{ij} = \eta^{ij}$

$\mathrm{div}\,T = 0 \qquad T^{ij} = (\rho + p)u^i u^j + p\eta^{ij}$

$\mathrm{Div}(nu) = 0 \qquad n = \frac{\#}{\mathrm{vol}} \qquad u^i = \frac{dx^i}{ds}$

diff wrt arclength $\Rightarrow$ unit

$$-ds^2 = -d(x^0)^2 + \sum_{i=1}^{3} d(x^i)^2 \qquad s = c\tau \quad x^0 = ct$$

$$\Rightarrow \; -1 = -(u^0)^2 + \sum_{i=1}^{3}(u^i)^2 \Rightarrow u^0 = \sqrt{1 + \left|\frac{v}{c}\right|^2}$$

$$v = \frac{dx^i}{dt}$$

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To take the classical limit:

so $e = \frac{\text{energy}}{\text{mass}}$

$$\rho = \underbrace{mnc^2}_{\equiv \rho_{CE}} + n\overset{m}{\swarrow}e$$

& view new mn as classical mass density

$$u^0 = \sqrt{1+\left(\frac{v}{c}\right)^2} = 1 + O\left(\frac{v}{c}\right)^2$$

$$u^i = \frac{v^i}{c}$$

$$\|u\|^2 = -1 \Rightarrow$$

$$|u^0| = \left|\frac{dx^0}{ds}\right| = \left|\frac{dt}{d\tau}\right| = \sqrt{1+\left(\frac{v}{c}\right)^2}$$

Plug into equations & get:

$$T^{\alpha\beta}{}_{,\beta} = \frac{1}{c}\left\{(nmc^2 + n\overset{m}{\swarrow}e + p)u^\alpha u^0 + p\eta^{\alpha 0}\right\}_{,t}$$

$$+ \left\{(\underbrace{nmc^2 + n\overset{m}{\swarrow}e}_{\rho} + p)u^\alpha \frac{v^j}{c} + p\eta^{\alpha j}\right\}_{,x^j}$$

Taking $\alpha = 0$ obtain:

$$T^{0\beta}{}_{,\beta} = \frac{1}{c}\Big[\Big\{(nmc^2+\overset{M}{n}e+p)\Big(1+\Big(\frac{v}{c}\Big)^2\Big)-p\Big\}_{\cancel{c}t}$$

$$+\Big\{(nmc^2+\overset{M}{n}e+p)\sqrt{1+\Big(\frac{v}{c}\Big)^2}\,\frac{v^j}{\cancel{c}}\Big\}_{x^j}\Big]$$

$$= \frac{1}{c}\Big[c^2\big\{(nm)_t+(nmv^j)_{x^j}\big\}$$

$$+\Big\{(\underbrace{nmv^2}_{\text{Looks like E but missing fact of } \frac{1}{2}!}+\overset{M}{n}e)_t + \Big(\big(\overset{n}{\tfrac{1}{2}}mv^2+\overset{M}{n}e+p\big)v^j\Big)_{x^j}\Big\} + O\Big(\frac{v}{c}\Big)^2\Big]$$

Taking $\alpha \equiv i = 1,2,3$

$$T^{i\beta}{}_{,\beta} = \frac{1}{c}\Big((nmc^2+\overset{M}{n}e+p)\sqrt{1+\Big(\frac{v}{c}\Big)^2}\,\frac{v^i}{c}\Big)_t$$

$$+\Big((nmc^2+\overset{M}{n}e+p)\frac{v^iv^j}{c^2}+p\delta^{ij}\Big)_{x^j}$$

$$= \big\{(nmv^i)_t+(nmv^iv^j+p\delta^{ij})_{x^j}\big\}+O\Big(\frac{v}{c}\Big)^2$$

For the number density -

$$(nu^B)_{,B} = \frac{1}{c}\Big(n\underbrace{\sqrt{1+\left(\tfrac{v}{c}\right)^2}}_{\sim 1+\frac{1}{2}\left(\frac{v}{c}\right)^2}\Big)_t + \left(n\frac{v^j}{c}\right)_{x^j}$$

$$= \frac{1}{c}\left\{n_t + (nv^j)_{x^j}\right\} + \frac{1}{c^3}\left(\frac{1}{2}nv^2\right)_t + O\left(\frac{1}{c}\left(\frac{v}{c}\right)^4\right)$$

$$\sqrt{1+\left(\tfrac{v}{c}\right)^2} = 1+\tfrac{1}{2}\left(\tfrac{v}{c}\right)^2 + O\left(\tfrac{v}{c}\right)^4$$

Collecting we obtain:

$$0 = T^{0B}{}_{,B} = c\left\{(nm)_t + (nmv^j)_{x^j}\right\} \qquad (1)$$

$$+ \frac{1}{c}\Big\{(\underbrace{nmv^2}_{\text{need } \frac{1}{2}mv^2} + ne)_t + \Big(\underbrace{\left(\tfrac{1}{2}mv^2 + ne + p\right)v^j}_{(E+P)\vec{v}}\Big)_{x^j}\Big\} + O\left(\frac{v}{c^3}\right)$$

$$0 = T^{iB}{}_{,B} = (nmv^i)_t + (nmv^iv^j + p\delta^{ij})_{x^j} + O\left(\frac{v}{c}\right)^2 \qquad (2)$$

$$0 = (nu^B)_{,B} = \frac{1}{c}\left\{n_t + (nv^j)_{x^j}\right\} + \frac{1}{c^3}\left(\frac{1}{2}nv^2\right)_t + O\left(\frac{1}{c}\left(\frac{v}{c}\right)^4\right) \qquad (3)$$

mult by mc^2 & subtract from (1)

To obtain CE equations –

- Mult (3) by mc to get

$$(mn)_t + (nmv^j)_{x^j} + \frac{1}{c^2}\left(\frac{1}{2}mnv^2\right)_t = O\left(\frac{v}{c}\right)^4 \qquad (4)$$

taking $\rho \equiv mn = \frac{\text{mass}}{\text{vol}}$ in CE, and sending $c \to \infty$ gives (MA)

$$\rho_t + \operatorname{div}(\rho\vec{v}) = 0$$

(eg now $\rho \equiv \rho_{CE}$ is classical mass density)

- Setting $\rho \equiv mn$ in (2) gives $(MO)^i$

$$(\rho v^i)_t + \operatorname{div}(\rho v^i \vec{v} + p e^i) = 0$$

- Most interesting part use (4) to substitute

$$(mn)_t + (nmv^j)_{x^j} = -\frac{1}{c^2}\left(\frac{1}{2}mnv^2\right)_t + O\left(\frac{v}{c}\right)^4 \qquad (5)$$

into (1) as follows:

Mult (5) by c^2 to get:

$$c^2\left\{(mn)_t + (mnv^j)_{x^j}\right\} + \tfrac{1}{2}mnv^2 = O\left(\frac{v}{c}\right)^2 \qquad (6)$$

Mult (1) by c to get:

$$c^2\left\{(mn)_t + (mnv^j)_{x^j}\right\}$$

$$+\left\{(\underbrace{nmv^2}_{\text{off by factor } \frac{1}{2}} + n\overset{m}{e})_t + ((\underbrace{\tfrac{1}{2}mv^2 + n\overset{m}{e}}_{E} + p)v^j)_{x^j}\right.$$

$$+ O\left(\frac{v}{c}\right)^2 = 0 \qquad (7)$$

Subtract (6) from (7)

$$\left\{(\overset{S_{cl}}{nmv^2} + n\overset{m}{e})_t + ((\tfrac{1}{2}mv^2 + n\overset{M}{e} + p)v^j)_{x^j} = O\left(\frac{v}{c}\right)^2\right.$$

$-\frac{1}{2}nmv^2$ from (6) ?

$c \to \infty$

$$\boxed{E_t + \operatorname{div}((E+P)\vec{v}) = 0}$$ (E...

- Conclude: It is pretty astounding how cons. of energy in the Newtonian sense is recoverd from Einstein's relativistic theory $\mathrm{Div}\left((\rho+p)u^\alpha u^\beta + p\,\eta^{\alpha\beta}\right)$

Indeed: a term from the number density equation that shows up due to the relativistic correction

$$u^0 = \sqrt{1+\left(\frac{v}{c}\right)^2} = 1 + \underbrace{\frac{1}{2}\left(\frac{v}{c}\right)^2}\, + \text{hot}$$

in the number density eqn, is just the deficit needed in the energy eqn to get $E_t + \mathrm{div}\left((E+p)\vec{v}\right) = 0$ after the leading order cons. of number cancels!!!

Molecules in random motion: (Derivation of Ideal Gas Law) ①

Assume local equilibrium so state determined by ρ, p, e, s, T

- $\langle \frac{1}{2} m \vec{v}^2 \rangle$ = ave KE of molecular motion
- <u>Assume</u> - Equipartition of energy - "Same amt of KE stored in all vibrational and translational modes equally"

$$\therefore \langle \tfrac{1}{2} m v_x^2 \rangle = \langle \tfrac{1}{2} m v_y^2 \rangle = \langle \tfrac{1}{2} m v_z^2 \rangle$$

$$\Rightarrow v^2 = |v|^2 = v_x^2 + v_y^2 + v_z^2 \Rightarrow \langle \tfrac{1}{2} m \vec{v}^2 \rangle = 3 \langle \tfrac{1}{2} m v_x^2 \rangle$$

- <u>Assume</u>: The ave KE stored in each vibrational & translational mode ~ temperature T ⇒

$$\langle \tfrac{1}{2} m v_x \rangle = \tfrac{1}{2} k T$$

- __Conclude__: $\langle \frac{1}{2} m \vec{v}^2 \rangle = 3 \langle \frac{1}{2} m v_x^2 \rangle = \frac{3}{2} kT$

- __Defn.__ The internal energy is energy stored in the vibrational modes, 3 for each atom in the molecule: two rotational & one radial $\Rightarrow$ $3r$ vibrational modes for r atoms in a molecule.

$r = 2$ for N_2 & O_2

Let U = internal energy in container in equilibrium $\Rightarrow$

$$U = N 3r \frac{1}{2} kT$$ = total internal energy

- Ideal Gas Law: $pV = NkT$ (macro)

$$\Leftrightarrow \boxed{p = nkT}$$

$n = \frac{\#}{\ }$, V = Vol, N = # of particles,

- Ie, ideal gas law says:

$$p = nkT = n\,\tfrac{2}{3}\left\langle \tfrac{1}{2} m \vec{v}^{\,2} \right\rangle = n \cdot 2 \left\langle \tfrac{1}{2} m v_x^2 \right\rangle$$

"factor of 2 because ave molecule bouncing off wall rebounds to deliver $2 \cdot \langle \tfrac{1}{2} m v_x^2 \rangle$ to create pressure force"

Conclude: Ideal Gas Law = Boyles Law reads

$$p = nkT$$

$$pV = NkT$$

Together with $U = N\,3r\,\tfrac{1}{2}kT$ gives

divide by M = tot mass

$$pV = \frac{2}{3r} U \iff pv = \frac{2}{3r} e$$

volume per mass = specific vol

int energy per mass = spec. int. energy

- $\dfrac{2}{3r} = \gamma - 1 \implies$

$$\boxed{pv = (\gamma - 1)e}$$

defines $\gamma - 1$

$\gamma \to 1$ as $r \to \infty$

$$e = \frac{N}{M}\frac{3r}{2}kT = \underbrace{\frac{N}{M(\gamma-1)}}_{C_v} kT = C_v T$$

This fully justifies our assumptions based on first principles — Last Time

<u>Theorem</u>: Assume

(1) $Pv = RT$ $\left(\text{From } Pv = (\gamma-1)e = \underbrace{(\gamma-1)C_v}_{R} T\right)$

(2) $e = C_v T$ $\left(C_v = \dfrac{R}{\gamma-1}\right)$

(3) $de = T\,ds - p\,dv$

⟹ Everything follows from Equipartition of energy alone!

Then:

$$e = \frac{c_v}{v^{\gamma-1}} e^{s/c_v} \equiv e(s,v)$$

$$p = -e_v(v,s) = C_v(\gamma-1)\frac{1}{v^{\gamma}}\exp\left(s/c_v\right)$$

Polytropic Eqn of State = gamma law gas

• <u>Thm</u>: $\gamma = \frac{C_p}{C_v}$ = "ratio of specific heats"

$\Rightarrow$ easy to measure.

Pf $C_p = \left.\frac{d(\text{Energy})}{dT}\right|_{p=\text{const}}$

$$\underbrace{dE}_{\substack{\text{chng}\\ \text{in tot}\\ \text{energy}}} \equiv \underbrace{de}_{\substack{\text{chng}\\ \text{in int}\\ \text{energy}}} + \underbrace{p\,dv}_{\substack{\text{work done}\\ \text{by compression}}} = C_v\,dT + p\,dv$$

$$C_p = \left.\frac{dE}{dT}\right|_{p=\text{const}} = C_v + p\left.\frac{dv}{dT}\right|_{p=\text{const}}$$

$pv = RT \Leftrightarrow v = \frac{RT}{p}$

$\left.\frac{dv}{dT}\right|_{p=\text{const}} = \frac{R}{p}$

$C_p = C_v + \cancel{p}\,\frac{R}{\cancel{p}}$

$= C_v + (\gamma - 1)\,C_v$

$\boxed{\gamma = \frac{C_p}{\ldots}}$