Connection

- II-translation: we wish to construct the notion of a vector "parallel" along a curve "C(s)" = "the non-rotating vectors of constant length carried with an observer in free fall"

- In a coord system x, look for a displacement rule of the form

\[ dy^i = - \Gamma^i_{jk} y^j dx^k \]  

\[ \uparrow \]

increment in coord

\[ y^i \] when \( y \) is II-translated

from \( x \) to \( x + dx \)

\[ \Gamma^i_{jk}(x) = \text{is given at each pt in each coord system} \]

that tell you how to construct II-translation in that coord system
Specifically: given $\Gamma^i_{jk}$, $Y^i_p = Y^i_p \frac{\partial x_i}{\partial x^j} \bigg|_p \in T_p M$ be a curve $c(s)$, $c(0) = p$ with $x$-coordinate representation $x(s) = x_0 c(s)$, we define the 1-translation $Y^i(s) \frac{d}{ds}$ along $x(s)$ as the soln of the ODE

\[
\begin{cases}
\frac{dy^i}{d\xi} = -\Gamma^i_{jk} y^j(s) \frac{dx^k}{d\xi} \\
y^i(0) = Y^i_p
\end{cases}
\]  

Here $x^i(s) = \frac{dx^i}{d\xi}$ so $x^i \frac{\partial}{\partial x^i} = \frac{dc}{d\xi}$ is the $x$-coordinate tangent to $c$.

Q: How must the $\Gamma^i_{jk}$ transform so that $Y^i(s)$ transforms like a vector $\in$ is indep of coordinates.
Theorem: \( Y^i (s) \) transforms like a vector \( \forall x, y \) iff \( \Gamma_{jk}^i \) satisfies the transformation rule (not a tensor):

\[
\Gamma_{\beta\sigma}^\alpha = \Gamma_{ij}^k \frac{\partial x^k}{\partial y^\beta} \frac{\partial x^\alpha}{\partial y^\sigma} + x^k \frac{\partial ^2 x^i}{\partial y^\beta \partial y^\sigma} \frac{\partial x^k}{\partial y^\sigma} \frac{\partial x^\alpha}{\partial y^\sigma} \tag{2}
\]

Proof: Let \( Y^d \approx x^a \), \( X^b \approx y^c \) be \( x, y \) in \( y \)-coordinates, so \( Y^i = y^a \frac{\partial x^i}{\partial y^a} \), \( X^i = x^a \frac{\partial x^i}{\partial y^a} \). Then \( \Theta \) is

\[
\frac{d}{ds} \left( y^a \frac{\partial x^i}{\partial y^a} \right) = - \Gamma_{ij}^k \frac{\partial x^k}{\partial y^b} \frac{\partial x^i}{\partial y^j} y^b x^a \tag{3}
\]
\[
\frac{d}{ds} \left( y^x \frac{\partial}{\partial y^x} \right) = \frac{d}{ds} y^x \frac{\partial}{\partial y^x} + \frac{\partial}{\partial y^x} \frac{\partial y^x}{\partial x^\alpha} \frac{\partial}{\partial x^\alpha} y^x
\]

Multiply thru by \((\frac{\partial}{\partial y^x})^{-1}\) gives

\[
\frac{d}{ds} y^x = \left\{ \Gamma^i_{\alpha \beta} \frac{\partial^{\alpha} x^i}{\partial y^\alpha} \frac{\partial}{\partial y^\alpha} + \frac{\partial y^x}{\partial x^i} \frac{\partial y^x}{\partial x^\alpha} \frac{\partial}{\partial x^\alpha} \right\} \frac{\partial}{\partial y^x}
\]

Gives (2)

Check: y^x mult thru by \(\frac{\partial}{\partial y^x}\) = y^x

Note: indep of \(\Gamma\) is

Conclude: The \(\Gamma\)'s transform like a (1,1) tensor with a correction term that is indep of \(\Gamma = \text{the same}\) for every connection.
Defn: We say $\Gamma$'s define a connection

$$\Gamma_{ij}^k \equiv \text{Christoffel Symbols (of 2nd kind)}$$

Cor 1: The difference between two connections is a tensor. \( \checkmark \) (Correction cancels out)

Cor 2: Given $\Gamma_{ij}^k$, $\Gamma_{ij}^k - \Gamma_{kj}^i$ transforms like a tensor. \( \checkmark \) (Correction cancels out)

Defn: $\Gamma_{ij}^k - \Gamma_{kj}^i = T_{ij}^k$ is the torsion tensor. (Measures "twist rel to nearby geodesics")

Cor 3: Symmetry $\Rightarrow \Gamma_{ij}^k = \Gamma_{kj}^i$ is a coord indept prop of connection.

Pf: $T_{ij}^k = 0$ in one coord syst $\Rightarrow 0$ in all

Cor 4: $\Gamma_{ijk}^j = 0 \Leftrightarrow$ in $x$-coord $\Rightarrow \Gamma$ symmetric (T_{ij}^k = 0)
\textbf{Corr 5}: If $\Gamma_{jk}^i(p) = 0$ in $x$-coordinates (= Locally Inertial coordinates) then $\Gamma$ symmetric & \( dy^i = 0 \mid_p \)
\[
\left. \frac{dy^i}{ds} \right|_p = 0, \quad X(Y)^i_p = 0 \quad \forall X \in T_p \mathcal{M}
\]
so \( y^i(s) = y^i(0) + O(s^2) \) = “parallel translation of vector fields to order $s^2$”

- \textbf{Assumption (Special Relativity)} In flat space, \( g_{ij} = \eta_{ij} \) & $\Gamma = 0$ in global Lorentz Frame

- \textbf{Assumption (General Relativity)} $\Gamma_{jk}^i(p) = 0$

in every locally inertial frame @ \( p \):

\( x \) locally inertial:
\[
\begin{cases}
  g_{ij}(p) = \eta_{ij}\\
  g_{ij,k}(p) = 0
\end{cases}
\]

\( \Rightarrow \bar{\Gamma}_{jk}^i \) for metric must be symmetric ($\bar{\Gamma}_{jk}^i = 0$)
Conclude: In principle, given g, one can obtain the metric connection of GR by constructing a locally inertial frame @ p, & using (*) to obtain $\Gamma^i_{ja}$ in any coord. syst.

Turn this around: You can get 1-trans. of $Y$ by locally inertial frames:

Cover c(i) by n pts spaced $O(\frac{1}{n})$ apart

11 translate $Y^i$ as constant in each coord syst
Transform $Y^i$ to (i+1)'s coord. system in overlap
Incure error $\Delta s$ in each $\Delta s$ step

$Y^i = Y_1^i + \sum_{j=1}^{n} \Delta s^j \rightarrow \sum_{i=0}^{n} \Delta s^2 = \Delta s \rightarrow 0$
Covariant Derivative: $\nabla_X Y$ defined by $\nabla_X Y = \lim_{\|\delta\| \to 0} \frac{Y_{\gamma}(\delta) - Y(\delta)}{\delta}$

- Given 2 vector fields $X, Y$
- Let $c_X(s)$ be integral curve of $X$ starting at $c_X(0) = p$, so $\frac{dc_X(s)}{ds} = X_{c_X(s)}$
- Let $Y(s) = Y_{c_X(s)}$
- $Y_{\gamma}(s)$ = lift-trans of $Y_p$ to $c_X(s)$ along $c_X$

Defn: $\nabla_X Y = \lim_{\|\delta\| \to 0} \frac{Y(\delta) - Y_{\gamma}(\delta)}{\delta}$

Since both $Y(s)$ & $Y_{\gamma}(s)$ are coord indep, this gives coord indep notion of deriv of vector field $Y$ in $X$ direction.

(only depends on $X(p)$?)
$\bigtriangledown_x Y$

$Y = Y''$

$Y_p = Y''$

$C_x(0) = p$

$(\bigtriangledown_x Y)_p = \lim_{s \to 0} \frac{Y(s) - Y_p(s)}{s}$

Both in $T_{C_x(s)} M$
The covariant derivative corrects vector differentiation to a tensor operation: i.e.,

\[
\nabla_x Y|_p = \lim_{s \to 0} \frac{Y(s) - Y(0)}{s} + \lim_{s \to 0} \frac{Y(0) - Y_i(5)}{s}
\]

\[
= X(Y) - \frac{dy^i}{ds} \frac{\partial}{\partial x^i}
\]

\[
= X(Y) + \Gamma^i_{jk} y^a X^k \frac{\partial}{\partial x^i}
\]

\[\text{coord indept but not a tensor}\]

\[\Gamma\text{ gives us a coord expression for a coord indept thing}\]

\[\text{We only have a coord way to express this limit?}\]

• In coordinates:

\[
(\nabla_x Y)^i = \frac{dy^i}{ds} - \frac{dy^i}{ds} = X^\sigma Y^\gamma_{\sigma} + \Gamma^i_{jk} y^\delta X^k
\]

\[\text{corrects } X(Y) \text{ to a tensor}\]
Conclude: $\nabla$ gives a second order expression to the $\Gamma$'s.

**Defn:** $Y$ is parallel along $c(s)$ if

$$\nabla_X Y = 0 \quad X = \frac{dc}{ds}$$

**Defn:** a curve $x(s)$ is a geodesic of $\Gamma$ if $X = \frac{dx}{ds}$ is parallel along $X$.

**Geodesic Equation:**

$$\nabla_X X = 0 \iff$$

$$\left(\nabla_X X\right)^2 = \frac{\partial^2 \gamma}{\partial x^i \partial x^j} X^j x^i + \Gamma^i_{jk} \frac{\partial x^j}{\partial s} \frac{\partial x^k}{\partial s} = 0$$

Since $x(s) = \dot{x}^i \implies x^i \frac{\partial}{\partial x^i} \dot{x}(s) = \ddot{x}^i(s)$

$$\implies \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$
Conclude: \( \nabla \) gives a coordinate independent expression to the \( \Gamma \)'s

Properties: \( (x_i \in T_p M, \ Y \text{ a vector field}) \)

1. \( \nabla_{ax_1 + bx_2} Y = a \nabla x_1 Y + b \nabla x_2 Y \) \( \{ \text{any smooth fn's} \ a, b : M \rightarrow \mathbb{R} \} \)

2. \( \nabla_x (Y_1 + Y_2) = \nabla_x Y_1 + \nabla_x Y_2 \)

3. \( \nabla_x [f(p) Y] = f(p) \nabla_x Y + X(f) \nabla_x Y \)

\( \text{Define} \ \nabla_x f = X(f) \ \text{so Liebniz rule holds} \)

4. \( \nabla_x Y - \nabla_y X = [X, Y] = L_x Y \) \( \{ \text{when } \Gamma^i_{jk} = \Gamma^i_{kj} \} \)

Proof: \( \nabla_x Y - \nabla_y X = X(Y) + \Gamma^i_{jk} Y^j X^k \)

\[ [X, Y] \underbrace{- Y(X)} = \Gamma^i_{jk} X^j Y^k \]

We assume symmetry here on
Extend $\nabla$ to covectors $\omega$ by requiring:

$$(\nabla_x \omega)(Y) = \nabla_x (\omega(Y)) \quad \forall \ Y \text{ s.t. } \nabla_x Y = 0$$

"so that $\nabla_x \omega = 0$ when $\omega(Y)$ evaluates parallel vector fields $Y$ along $c_x(s)$ as constant."

That is: $\nabla_x \omega = (\nabla_x \omega)_0 \cdot d \dot{\sigma}$

so $$(\nabla_x \omega)(Y) = (\nabla_x \omega)_0 \cdot Y$$

subject to $$(\nabla_x \omega)_0 \cdot Y = \nabla_x (\omega(Y))$$ when $\nabla_x Y = 0$. 
So assume $\nabla_x y = 0$, and calculate

$$\nabla_x \omega(y) = X(\omega(y)) = X^i \frac{\partial}{\partial x^i} \left( \omega^\sigma \gamma^\sigma \right)$$

$$= X^i \left( \frac{\partial}{\partial x^i} \omega^\sigma \right) \gamma^\sigma + X^i \omega^\sigma \frac{\partial}{\partial x^i} (\gamma^\sigma)$$

$$= \left( X^i \frac{\partial}{\partial x^i} \omega^\sigma \right) \gamma^\sigma + \omega^\sigma \left( (\nabla_x \gamma^\sigma) - \Gamma^\sigma_{\alpha\beta} X^\alpha X^\beta \right)$$

$$= \left( X^i \frac{\partial}{\partial x^i} \omega^\sigma - \Gamma^\sigma_{\alpha\beta} \omega^\alpha \gamma^\beta \right) \gamma^\sigma$$

$$= (\nabla_x \omega)_\sigma \gamma^\sigma$$

$$\Rightarrow (\nabla_x \omega)_\sigma = X^i \omega^\sigma_{,i} - \Gamma^\sigma_{\sigma\kappa} \omega^\kappa X^\kappa$$

We can also write

$$\nabla_i y = \nabla_x \frac{\partial}{\partial x^i} \gamma^\sigma = \left( \gamma^\sigma_{,i} + \Gamma^\sigma_{ij} \gamma^j \right) \frac{\partial}{\partial x^\sigma}$$

$$= \gamma^\sigma_{,i} \frac{\partial}{\partial x^\sigma}$$

$$\nabla_i \omega = \omega^\sigma_{,i} \frac{\partial}{\partial x^\sigma} = (\omega^\sigma_{,i} - \Gamma^\sigma_{\sigma\iota} \omega^\iota) \frac{\partial}{\partial x^\sigma}$$
we can extend $\nabla$ to arb. tensor fields by asking:

$$\left[ \nabla_x T \right] (x_1, \ldots, x_m, \omega^1, \ldots, \omega^k)$$

$$= \nabla_x \left[ T (x_1, \ldots, x_m, \omega^1, \ldots, \omega^k) \right]$$

a scalar

for all $x_1, \ldots, x_m, \omega^1, \ldots, \omega^m$ not along $C_x(\xi)$.

Formula:

$$\nabla_x \left( T_{i, j} \, dx^j \otimes \frac{\partial}{\partial x_i} \right) \text{ has components}$$

$$X^k \frac{\partial}{\partial x^i} T_{j, k} + \sum_{\sigma \tau} \Gamma^i_{\sigma \tau} T^{\sigma}_{j, \tau} X^\tau - \sum_{\sigma \tau} \Gamma_{j, \tau}^{\sigma} T^i_{\sigma, \tau} X^\tau = (\nabla_x T)^i_j$$

a term for every contravariant index$^{\uparrow}$

a term for every covariant index$^{\uparrow}$
Defn: we let $\nabla T$ denote the (tensor) with components $T_{\dot{a} \cdots \dot{a}}^{\dot{a} \cdots \dot{a}}$ when $T$ has components $T_{\dot{a} \cdots \dot{a}}^{\dot{a} \cdots \dot{a}}$.

$$(\nabla X)^i_j = X^i_j + \Gamma^i_{\sigma j} X^\sigma \text{ etc.}$$

Properties:

1. $\nabla T$ is a tensor for any tensor $T$.

2. $\nabla_X (A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B$

3. $\nabla_X (T^i_j) = (\nabla_X T)^i_j$

More generally, $\nabla_X$ commutes with contraction.

Ref MTW pg 223
geodesics of a connection:

**Defn:** a geodesic is a curve along which the tangent vector is parallel.

**Equn** for geodesic of connection:

Let \( x(s) \) be the \( x \)-coordinate curve with tangent vector \( \frac{dc_x}{ds} = X \)

ie., \[
\frac{dx^i}{ds} = X^i.
\]

then in \( x \)-coordinate, the condition \( \nabla_x X = 0 \) ⇒

\[
(\nabla_x X)^i = X^\sigma \frac{\partial}{\partial x^\sigma} X^i + \Gamma^i_{\sigma \tau} X^\sigma X^\tau
\]

\[
= \frac{d}{ds} x^i(s) + \Gamma^i_{\sigma \tau} X^\sigma X^\tau
\]

\[
= \frac{d^2 x^i}{ds^2} + \Gamma^i_{\sigma \tau} \dot{x}^\sigma \dot{x}^\tau = 0
\]

**Sln** i.e.

\[
x^i(0) = x^i_0
\]

\[
\dot{x}^i(0) = v^i
\]
Note: if $x(s)$ solves (**) with initial vector $x$, then $x(cs)$ solves (**) with initial vector $cX$. 
Thm: Two symmetric connections having the same geodesics agree. Geodesics for $\tilde{\Gamma}^i_{\alpha\beta}$ agree with geodesics for $\frac{1}{2}(\Gamma^i_{\alpha\beta} + \tilde{\Gamma}^i_{\beta\alpha})$.

Pf: Let $\Gamma$, $\tilde{\Gamma}$ two connections $\Rightarrow$ in $x$-coord.:

$$\frac{d^2 x^i}{ds^2} + \tilde{\Gamma}^i_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \iff \frac{d^2 x^i}{ds^2} + \Gamma^i_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

$x^i(0) = x^i_0$ same i.c.'s.

$$\dot{x}^i(0) = \dot{x}^i_0$$

choose: $\dot{x}^i_0 = \{ 1 \text{ if } i = \alpha \}$ fix $x^i_0 = x(\rho)$

Then at $p$:

$$\Rightarrow \frac{d^2 x^i}{ds^2} = -\tilde{\Gamma}^i_{\alpha\beta} (\dot{x}^\alpha_0)^2 = -\Gamma^i_{\alpha\beta} (\dot{x}^\alpha_0)^2$$

$$\Rightarrow \tilde{\Gamma}^i_{\alpha\beta} = \Gamma^i_{\alpha\beta} \text{ all } \alpha$$

$$\dot{x}^i_0 = \{ 1 \text{ if } i = \alpha \}, \text{ i.e. } x^i_0 = \{ 0 \text{ o.w.} \}$$

$$\Rightarrow \frac{d^2 x^i}{ds^2} = -\tilde{\Gamma}^i_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta - \Gamma^i_{\beta\alpha} \dot{x}^\beta \dot{x}^\alpha = -\Gamma^i_{\beta\alpha} \dot{x}^\beta \dot{x}^\alpha - \Gamma^i_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$$

$$\Rightarrow \frac{1}{2}(\tilde{\Gamma}^i_{\alpha\beta} + \Gamma^i_{\beta\alpha}) = \frac{1}{2}(\Gamma^i_{\alpha\beta} + \Gamma^i_{\beta\alpha})$$
Conclude: Symmetry $\Rightarrow$ $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$

Non-symmetry $\Rightarrow$ geodesics for $\tilde{\Gamma}$ are the same as the geodesics for the symmetric connection $\Gamma^i_{\alpha\beta} = \frac{1}{2}(\Gamma^i_{\alpha\beta} + \Gamma^i_{\beta\alpha})$.

Assume $\Gamma$ is the connection for metric $g$ at $p$.

In a local Lorentz frame, the geodesics for $g$, $\Gamma(p) = 0$, to 1st order, their geodesics to agree:

Theorem:
Theorem: (The connection that goes with metric $g$)

Assume $g$ is a Lorentzian metric with signature $\gamma = \text{diag}(-1,1,1,1)$. Let $\Gamma$ be a connection whose components vanish in a coordinate frame where (conditions for local inertial frame)

\[ g_{ij}(p) = \gamma \]

\[ g_{ij,k}(p) = 0. \]

Then in any other coordinate frame $\gamma$ we must have

\[ \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma} \left\{ -g_{\alpha\beta,\sigma} + g_{\gamma,\alpha\beta} + g_{\beta,\gamma\alpha} \right\} \]

thus giving a formula for the Christoffel symbols for a unique symmetric connection determined by the metric $g$. 

Proof: The transformation law for a connection is given by
\[ \Gamma_{\rho \beta}^\gamma = \Gamma_{\alpha \beta}^\gamma \frac{dx^\alpha}{dx^\gamma} \frac{dx^\rho}{dx^\gamma} + \frac{\partial x^\alpha}{\partial y^\rho} \frac{\partial y^\beta}{\partial y^\gamma} \frac{\partial y^\gamma}{\partial x^\gamma} \] (vi)

Since \( \Gamma_{\beta \alpha}^\gamma = 0 \) at \( p \), we have
\[ \Gamma_{\rho \beta}^\gamma = \Gamma_{\beta \rho}^\gamma (p) = \frac{\partial^2 x^i}{\partial y^\rho \partial y^\beta} \frac{\partial y^\gamma}{\partial x^i}. \] (vii)

Now the components of the metric \( g_{\alpha \beta} \) are given by
\[ g_{\alpha \beta} = \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^i}{\partial y^\beta}, \] (vi)

with \( g = J^t \eta J \) (matrix transformation) (vii)
Differentiating (vi) and using (11) gives

\[ g_{\alpha\beta,\gamma}(t) = \eta_{ij} \frac{\partial^2 x^i}{\partial \theta^j} + \eta_{ij} \frac{\partial x^i}{\partial \theta^j} \frac{\partial x^j}{\partial \theta^k} \]

Now

\[ \Delta_{\alpha\beta,\gamma} \quad \Delta_{\beta\gamma,\alpha} \]

\[ \text{Notation} \]

So

\[ \Delta_{\alpha\beta,\gamma} = \Delta_{\beta\gamma,\alpha} \] (equality of mixed partials)

Therefore,

\[ -g_{\alpha\beta,\gamma} + g_{\delta\gamma,\alpha} + g_{\beta\delta,\gamma} = -\Delta_{\alpha\beta,\gamma} - \Delta_{\delta\gamma,\alpha} \]

\[ + \Delta_{\delta\beta,\alpha} + \Delta_{\alpha\delta,\beta} \]

\[ + \Delta_{\beta\delta,\gamma} + \Delta_{\delta\alpha,\beta} \]

\[ = 2 \Delta_{\delta\beta,\gamma} = 2 \eta_{ij} \frac{\partial^2 x^i}{\partial \theta^j} \frac{\partial x^j}{\partial \theta^k} \]
Now consider:

\[
(g_{r\sigma}) = (g_{ab})^{-1} = (J^T \eta J)^{-1} = J^{-1} \eta (J^T)^{-1}
\]

Consider now the expression

\[
\frac{1}{2} g^{r\sigma} \left\{ -g_{ab,\sigma} + g_{a,\sigma b} + g_{b,\sigma a} \right\}
\]

\[
= g^{r\sigma} \left\{ \epsilon_{ij} \frac{\partial^2 x^i}{\partial y^a \partial y^b} \frac{\partial x^i}{\partial y^\sigma} \right\}
\]

\[
= J^{-1} \gamma (J^T)^{-1} \quad J^{tr} \eta M_b \quad M_b = \frac{\partial^2 x^j}{\partial y^a \partial y^b}
\]

\[
= J^{-1} \gamma (J^T)^{-1} \quad J^{tr} \eta M_b = J^{-1} M_b
\]

\[
= \frac{\partial x^b}{\partial x^i} \frac{\partial^2 x^i}{\partial y^a \partial y^b} = \Gamma^r_{a b}
\]
Corollary 10: The geodesics for metric $g$ satisfy the geodesic equation, $\Gamma$ given in (ii):

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$  \hspace{1cm} (60)

Theorem: Curves that satisfy (60) are critical points of the $(\text{length})^2$ (Do not hold separately)

Length $\mathcal{L}[c(\cdot)] = \int_{\xi_1}^{\xi_2} \| \dot{x}(s) \|^2 \, ds$

$$= \int_{\xi_1}^{\xi_2} \langle x, \dot{x} \rangle \, ds = \int_{\xi_1}^{\xi_2} g_{ij}(s) \dot{x}^i \dot{x}^j \, ds$$

$\Rightarrow$ The Euler-Lagrange equations are equivalent to

Ref: Spivak, Dubrovin, Novikov

Modern Geom I
 Cov (2). Let $T^\xi_{\alpha\beta}$ be any torsion tensor,

$$T^\xi_{\alpha\beta} = \Gamma^\xi_{\alpha\beta} - \Gamma^\xi_{\beta\alpha}$$

so $T^\xi_{\alpha\beta}$ is any antisymmetric tensor. Then if $\Gamma^\xi_{\alpha\beta}$ is given by (iii), it follows that

$$\Gamma^\xi_{\alpha\beta} = \Gamma^\xi_{\alpha\beta} + T^\xi_{\alpha\beta}$$

has the same geodesics as $\Gamma^\xi$, but is not symmetric.

We could show: non-symmetry introduces a twist relative to nearby geodesics.

Cor (3) $\exists!$ symmetric connection whose geodesics are geodesics of the metric.
Lemma: Let \( g \) have connection \( \Gamma \leftrightarrow \nabla \).

Then

1. \( \nabla_z g = 0 \) \( \forall z \)

and

2. \( \nabla_z \left( g_{ij}(x^k) \right) = g_{ij}(\nabla_z x^k)^2 x^k + g_{ij} x^k (\nabla_z x^k)^3 \)

Note: \( 2 \Rightarrow \) angles \& lengths are preserved under \( \Gamma \rightarrow \) translation.

Proof: For 1, go to a locally Covariant coord frame where \( g_{ij;k} = 0 \) \& \( \Gamma_{ij}^k = 0 \).

Formula for cov. deriv \( \Rightarrow \nabla_z g = 0 \).

\( \nabla_z g \) a tensor \( \Rightarrow \nabla_z g = 0 \) in all coords.
Corollary: For metric connection $\nabla$ for $g$:

1. $\nabla_z \langle X, Y \rangle = \langle \nabla_z X, Y \rangle + \langle X, \nabla_z Y \rangle$

2. $\nabla_z (g_{\alpha i} X^\alpha dx^i) = g_{\alpha i} \nabla_z (X^\alpha \frac{\partial}{\partial x^i})$

Note generally - $\nabla_z$ commutes with raising & lowering of indices (F10)

Note:

$\Rightarrow$ if $X, Y$ are parallel along $C_X(\mathbb{S})$, then $\nabla_z \langle X, Y \rangle = Z(\langle X, Y \rangle) = 0 \Rightarrow$

$\Rightarrow$ translation preserves lengths & angles between vectors.
\[ \nabla_2 T = \nabla_2 (g \otimes X \otimes Y) + g \otimes \nabla_2 X \otimes Y + g \otimes X \otimes \nabla_2 Y \]

But: \( \nabla_2 (a \text{ contraction of } T) = \text{ contraction of } (\nabla_2 T) \)

LHS: \( \nabla_2 \left( g_{ij} x^k y^l \, dx^i \otimes dx^j \otimes \frac{2}{\partial x^a} \otimes \frac{2}{\partial x^b} \right) \) contract twice
\( \Rightarrow \nabla_2 (g_{ij} x^i y^j) = \nabla_2 \langle x_j y \rangle \)

RHS: \( g_{ij} (\nabla_2 x)^k y^l \, dx^i \otimes dx^j \otimes \frac{2}{\partial x^a} \otimes \frac{2}{\partial x^b} \) contract twice
\( \Rightarrow g_{ij} (\nabla_2 x)^i y^j = \langle \nabla_2 x_j y \rangle \)

Sim: contract: \( g \otimes X \otimes \nabla_2 Y \rightarrow \langle x_j \nabla_2 Y \rangle \)
**Corollary**: along a geodesic, \( \langle x, x \rangle \)
is constant (i.e., \( \nabla_x x = 0 \) \( \Rightarrow \))
timelike geodesics remain timelike
spacelike " " spacelike
null " " null.

**Assmt**: Free-fall paths are timelike geodesics
glitch rays follow null geodesics.
Let $\Gamma$ be any connection & $\nabla$ its covariant derivative. Then

$$(LyT)^i_j = \left( \frac{d}{ds} \Phi_s^* T_{\Phi_s(p)} \bigg|_{s=0} \right)^i_j$$

$$= (\nabla_y T)^i_j - T^i_\sigma (\nabla_{\sigma} Y)^i_j + T^i_\sigma \left( \nabla_\sigma Y \right)^\sigma_j \quad \text{(7)}$$

Corr:

$$Cov: (Lyg)_{ij} = (\nabla_y g)_{ij} + g_{\sigma j} \nabla_i Y^\sigma + g_{i \sigma} \nabla_\sigma Y^\sigma$$

$$= \nabla_i Y_j + \nabla_j Y_i = Y_{j;i} + Y_{i,j}$$
Pf. of theorem: \( L_Y f = Y(f) \); \( L_Y X = [Y, X] \)

\[
L_Y (w_\sigma X^\sigma) = Y (w_\sigma X^\sigma) \quad \text{apply to every index } \Gamma^{-\sigma}...
\]

Also

\[
L_Y (w_\sigma X^\sigma) = (L_Y w)_\sigma X^\sigma + w_\sigma (L_Y X)^\sigma
\]

\[
= (L_Y w)_\sigma X^\sigma + w_\sigma [Y, X]^\sigma \quad (A)
\]

\[
= (L_Y w)_\sigma X^\sigma + w_\sigma (\nabla_y X)^\sigma - (\nabla_x Y)^\sigma
\]

\[
Y (w_\sigma X^\sigma) = (\nabla_y w)_\sigma X^\sigma + w_\sigma (\nabla_y X)^\sigma \quad (B)
\]

Equating (A) & (B):

\[
(\nabla_y w)_\sigma X^\sigma = (L_Y w)_\sigma X^\sigma = w_\sigma (\nabla_x Y)^\sigma
\]

\[
X = \frac{\partial}{\partial x_i} : \quad (y^w)_i = (\nabla_y w)_i + w_\sigma \frac{\partial}{\partial x_i} Y^\sigma
\]
**Divergence**: Given a vector field $\mathbf{X}$, define
\[
\text{div} \mathbf{X} = X^i ;_i \quad \text{a scalar}.
\]

- In local Lorentz coordinates, $\text{div} \mathbf{X} = X^i ;_i = \text{classical divergence}$.

**Theorem**: $\text{div} \mathbf{X} = \frac{1}{\sqrt{-g}} \left( X^i \sqrt{-g} \right) ;_i$

**Proof**:
\[
X^i ;_i = X^i ,_i + \Gamma^i_{\alpha i} X^\alpha
\]

We evaluate $\Gamma^i_{\alpha i} = \Gamma^i_{i \alpha}$.

We know from $\nabla g = 0$ that
\[
0 = g_{ij, k} - \Gamma^\sigma_{ik} g_{\sigma j} - \Gamma^\sigma_{jk} g_{\sigma i}
\]

Mult by $g^{ij}$ & contract:
\[
0 = g^{ij} g_{ij, k} - \Gamma^i_{\alpha k} g^{\alpha j} - \Gamma^j_{\alpha k} g^{\alpha i} \Rightarrow \Gamma^i_{\alpha k} = \frac{1}{2} g^{ij} g_{ij, k}
\]

i.e., $g_{ij} g^{ij} = \delta^2 \omega$.
But: \[ g^{ij} = \frac{1}{g} \Delta^{ij} \]

Matrix of cofactors: \[ \Delta^{ij} = (\det g)_{ij} \]

\[ \Delta = \frac{\partial g}{\partial g_{ij}} \] just expand \( g \) by \( i \)th color.

\[ \Rightarrow \Gamma^i_{jk} = \frac{1}{2} g^{ij} g_{ij} \n = \frac{1}{2} \frac{\partial g}{\partial x^k} g_{ij} \]

\[ = \frac{\partial}{\partial x^k} \log \sqrt{-g} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} \sqrt{-g} \]

Thus: \[ \Gamma^i_{jk} = \frac{\partial}{\partial x^k} \log \sqrt{-g} \]

and

\[ \text{div} X = X^i, i + \frac{1}{\sqrt{-g}} \left( \frac{\partial}{\partial x^k} \sqrt{-g} \right) X^k \]

\[ = \frac{1}{\sqrt{-g}} \left( X^i \sqrt{-g} \right), i \]
**Gauss Theorem**: A coordinate independent version of div theorem that holds for vector fields:

**Theorem**: \( F = F^i \frac{\partial}{\partial x^i} \) a vector field. Then

\[
\int_{\Omega} \text{div} F \, \eta = \int_{\partial \Omega} \frac{\langle F, \eta \rangle}{\langle n, n \rangle} \eta \cdot n = \int_{\partial \Omega} F^\alpha d\Sigma_\alpha
\]

Here: \( \eta = \sqrt{-g} \) is volume form

\( \Omega \) is 4-dimensional region

\( \text{div} F \equiv F^\sigma_{\sigma} \) a scalar

\( n \) is unit outward normal to \( \partial \Omega \), normal relative to metric \( g, \langle \rangle \).

\( d\Sigma_\alpha = \eta^{-1} \frac{\partial}{\partial x^\alpha} \)
Proof: By previous result,

\[ F \frac{\partial}{\partial x} = \frac{1}{\sqrt{1-g}} (F \sqrt{1-g}), \]

Assume \( \Sigma \) lies in a single coord chart: \( x \).

Then the classical divergence Thm holds as follows:

\[ \int_{\Sigma} F = \int_{\Sigma} F \sqrt{1-g} = \int_{\Sigma} \frac{1}{\sqrt{1-g}} (F \sqrt{1-g}) \frac{\partial}{\partial x} \]

\[ = \int_{\Sigma} \left( F \sqrt{1-g} \right) \frac{\partial}{\partial x} \]

\[ = \int_{\Sigma} \left( F \frac{\partial}{\partial x} \right) \]

where \( \hat{n} \) is the coordinate unit normal to \( \Sigma \).
Now: if \( \vec{n} \) is the Euclidean coordinate normal and \( n = n^a \varepsilon^a \) is the metric unit normal, then \( \vec{n} \cdot \vec{n} = n_i \cdot n_i = \text{Eg} \)

\[
\hat{n} = \sum \varepsilon \cdot \hat{F} = \int \sqrt{g} \varepsilon \cdot \hat{F} \cdot n_i \hat{n} = \int \sqrt{g} \varepsilon \cdot \{ (F \cdot \hat{n}) \hat{n} + F - (F \cdot \hat{n}) \hat{n} \} \in T_{\partial n} \Omega
\]

\[
\text{no contribution for } x_1 \cdots x_3 \in T\Omega
\]

\[
d \Sigma_x = \eta \cdot \frac{\Sigma}{x^x}
\]
But we can also write

\[ \int \limits_{\mathcal{L} \in \mathcal{G}} F = \int \limits_{\mathcal{L} \in \mathcal{G}} \left\{ \frac{\langle F, n \rangle}{\langle n, n \rangle} n + F - \frac{\langle F, n \rangle}{\langle n, n \rangle} n \right\} \]

in \mathcal{L} \in \mathcal{G}

\[ = \int \limits_{\mathcal{L} \in \mathcal{G}} \left\{ \frac{\langle F, n \rangle}{\langle n, n \rangle} n \right\} = \int \frac{\langle F, n \rangle}{\langle n, n \rangle} n \ \mathcal{L} \in \mathcal{G} \]

"metric version of "
"F on n"

"metric version of "
"ds"
**Application:** Cons. of mass principle

**Ex.:** \( S(x) = \frac{\text{mass}}{3\text{-vol}} \) in rest frame \( \approx \) dust.

\( u = \frac{dx_i}{ds} \) 4-velocity of particles

\[
ds = \sqrt{(dx^0)^2 - |dx|^2} = \sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}} \, dx^0
\]

\[
\frac{dx^0}{ds} = \frac{1}{\sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}} = \text{Lorentz contraction factor}
\]

\[ x^0 \quad u \quad \mathbf{v} \quad x^i = \frac{\text{mass}}{3\text{-vol}} \]

\[
\mathbf{u} = \frac{\text{mass}}{3\text{-vol}} \frac{dx^i}{ds} \frac{dx^0}{dx^0} = \left\{ \mathbf{u}, \frac{2}{3} \mathbf{x}^0 \right\}
\]

\[
\frac{\text{mass}}{\text{area } dx^0} \text{ in } x\text{-frame} = \frac{\text{mass}}{3\text{-vol}} \frac{dx^i}{ds} \frac{dx^0}{dx^0} = \left\{ \mathbf{u}, \frac{2}{3} \mathbf{x}^1 \right\}
\]
Conclude: At each pt. on $\Sigma$, 

$$\langle \mathbf{s}_n \cdot \mathbf{n} \rangle = \frac{\text{mass}}{\text{metric 3-vol}} \equiv \begin{cases} \text{density if } n \text{ time-like} \\ \text{mass flux if } n \text{ space-like} \end{cases}$$

coordinate independent meaning

$\therefore \quad \langle \mathbf{s}_n \cdot \mathbf{n} \rangle \eta \mathbf{n} (x_1, x_2, x_3)$

evaluates the total mass in 3-hyperplane $X_1 X_2 X_3$ (per metric vol)

$$\int \langle \mathbf{s}_n \cdot \mathbf{n} \rangle \eta \mathbf{n} = \int \frac{\langle \mathbf{s}_n \cdot \mathbf{n} \rangle}{\langle \mathbf{n} \cdot \mathbf{n} \rangle} \eta \mathbf{n} \left( \frac{2}{\partial x^1} \frac{2}{\partial x^2} \frac{2}{\partial x^3} \right) \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$$

$\Omega_x$

$\Sigma_1$ = "total mass at $\Sigma_1$"

$\Sigma_2$ = "total mass at $\Sigma_2$"

$\Sigma_3\Sigma_4$ = "total passing out of $\Omega$ through $\Sigma_3\Sigma_4$"
Global coord indept statement of Cons. of mass is

\[ 0 = \int <su, n> \eta \nabla \eta - \int \text{div} u = 0. \]

\[ \nabla \cdot u = 0. \]

Note: since \( \int <su, n> \eta \nabla \eta = \int su^2 d\Sigma_i \),
we can think of \( su^2 d\Sigma_i \) as "eating 3-vol's \( X_1 \wedge X_2 \wedge X_3 \) & evaluating the total mass in the \( \eta \) = mass flux if one of the \( X_i \) is timelike."

Ref MTW.
For energy & momentum:

\[ p \frac{dx^0}{ds} \] \( u \) energy flux vector

\[ p \frac{dx^i}{ds} \] \( u_i \) momentum flux vector

Extra factor of \( \frac{dx^0}{ds} \) needed because energy increases with velocity due to Lorentz Contra effect + indicare in KE.

Components of tensor: \( T^{ij} = p u^i u^j \)

Conclude: \( F = p \frac{dx^0}{ds} u \) satisfy global cons law

\[ \int \frac{\langle F, n \rangle}{\langle n, n \rangle} \, n = 0 \tag{K} \]

Only so long as \( F \) transforms like a vector. It does under linear coord
transformations \Rightarrow \text{global inv. cons. law holds in Special Relativity when we restrict to local Lorentz transformations.}

In general relativity, we can only require local conservation of energy:

\[ \text{div } T = 0 = T_{;i}^i \text{.} \]

This reproduces (x) in flat space & approximately reproduces (x) in local Lorentz frames where \( g_{ij} = \eta_{ij} + O(1/x^2) \).