Connection

- ∥-translation: we wish to construct the notion of a vector "parallel" along a curve "$c(s)$" ≡ "the non-rotating vectors of constant length carried with an observer in freefall"

- In a coord system x, look for a displacement rule of the form

$$dy^i = -\Gamma^i_{jk}\, y^j\, dx^k \qquad (1)$$

increment in coord y^i when Y is ∥-translated from x to $x+dx$

$\Gamma^i_{jk}(p)$ = #'s given at each pt in each coord system that tell you how to construct ∥-translation in that coord. system

• Specifically: given Γ^i_{jk}, $Y_p = Y^i \frac{\partial}{\partial x^i}\Big|_p \in T_p M$ & a curve $c(\xi)$, $c(0) = p$ with x-coord representation $x(\xi) = \underset{\sim}{x} \circ c(\xi)$, we define the ‖-translation $Y^i(\xi) \frac{\partial}{\partial x^i}$ along $x(\xi)$ as the soln of the ODE

$$\begin{cases} \dfrac{dY^i}{d\xi} = - \Gamma^i_{jk}\, Y^j(\xi) \dfrac{dx^k}{d\xi} \\ Y^i(0) = Y^i_p \end{cases} \qquad (*)$$

Here $\dot{x}^i(\xi) = \frac{dx^i}{d\xi} = X^i$ so $X^i \frac{\partial}{\partial x^i} = \frac{dc}{d\xi}$ is the $\underset{\sim}{x}$-coord tangent to c.

• Q: How must the Γ^i_{jk} transform so that $Y^i(\xi)$ transforms like a vector $\equiv$ is indep't of coordinates.

<u>Theorem</u>: $y^i(\xi)$ transforms like a vector $\forall X, Y$ iff Γ^i_{jk} satisfies the transformation rule (<u>not</u> a tensor):

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{i}_{jk} \frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i} + \frac{\partial^2 x^i}{\partial y^\beta \partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i} \qquad (2)$$

<u>Proof</u>: Let $Y^\alpha \frac{\partial}{\partial y^\alpha}$, $X^\beta \frac{\partial}{\partial y^\beta}$ be X, Y in $\underset{\sim}{y}$-coords, so $Y^i = Y^\alpha \frac{\partial x^i}{\partial y^\alpha}$, $X^i = X^\alpha \frac{\partial x^i}{\partial y^\alpha}$.

Then (*) is

$$\underbrace{\frac{d}{d\xi}\left(Y^\alpha \frac{\partial x^i}{\partial y^\alpha}\right)}_{\parallel} = -\Gamma^i_{jk}\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma} Y^\beta X^\gamma \qquad (3)$$

chng summation to γ

$$\left(\frac{d}{d\xi}Y^\alpha\right)\frac{\partial x^i}{\partial y^\alpha} + \underbrace{Y^\alpha \frac{d}{d\xi}\frac{\partial x^i}{\partial y^\alpha}}_{Y^\gamma X^\beta \frac{\partial^2 x^i}{\partial y^\beta \partial y^\gamma}}$$

so

$$\frac{d}{d\xi}\left(Y^\alpha \frac{\partial x^i}{\partial y^\alpha}\right) = \frac{dY^\alpha}{d\xi}\frac{\partial x^i}{\partial y^\alpha} + \frac{\partial y^\alpha}{\partial x^j}\frac{\partial^2 x^j}{\partial y^\beta \partial y^\gamma} X^\beta Y^\gamma$$

Multiply(3) thru by $\left(\frac{\partial x^i}{\partial y^\alpha}\right)^{-1}$ gives

$$\frac{d}{d\xi} Y^\alpha = -\Big\{ \underbrace{\Gamma^i_{jk}\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i} + \frac{\partial y^\alpha}{\partial x^j}\frac{\partial^2 x^j}{\partial y^\beta \partial y^\gamma}}_{\Gamma^\alpha_{\beta\gamma}} \Big\} X^\beta Y^\gamma$$

giving (2)

check: $Y^\alpha \frac{\partial x^i}{\partial y^\alpha} = \mathbf{Y}^i$ "

mult thru by $\frac{\partial y^\alpha}{\partial x^i}$ $\left(\frac{\partial x^i}{\partial y^\alpha}\right)^{-1}$ "

$= Y^\alpha = \mathbf{Y}^i$

Note: indept of Γ's

<u>Conclude</u>: The Γ's transform like a $\binom{1}{2}$ tensor with a correction term that is indept of Γ = the <u>same</u> for every connection.

- **Defn**: We say Γ's define a connection

$\Gamma^i_{jk} \equiv$ Christoffel Symbols (of 2nd kind)

Cor ① The difference betw two connections is a tensor. ✓ ("Correction cancels out")

Cor ② Given Γ^i_{jk}, $\Gamma^i_{jk} - \Gamma^i_{kj}$ transforms like a tensor ✓ ("Correction cancels out")

Defn: $\Gamma^i_{jk} - \Gamma^i_{kj} \equiv T^i_{jk}$ is the torsion tensor. (Measures "twist rel to nearby geodesics")

Cor ③ Symmetry $\equiv$ "$\Gamma^i_{jk} = \Gamma^i_{kj}$" is a coord indept prop of connections

Pf. $T^i_{jk} = 0$ in one coord syst $\Rightarrow 0$ in all. ✓

Cor ④ $\Gamma^{i}_{jk} = 0$ @ p in x-coords $\Rightarrow$ Γ symmetric ($T^i_{jk} = 0$!)

Cor ⑤ : If $\Gamma^i_{jk}(p) = 0$ in $\underset{\sim}{x}$-coords (≡ Locally Inertial coordinates) then Γ symmetric & $dy^i = 0 \big|_p$

$$\left.\frac{dy^i}{d\xi}\right|_p = 0, \quad = X(Y)^i_p = 0 \quad \forall X \in T_pM$$

so $y^i(\xi) = y^i(0) + O(\xi^2)$ ≡ "parallel translation keeps comp's constant to order ξ^2"

(p, $|\text{dist from } p|^2$)

- **Assumption** (Special Relativity) In flat space, $g_{ij} = \eta_{ij}$ & $\Gamma \equiv 0$ in global Lorentz Frame

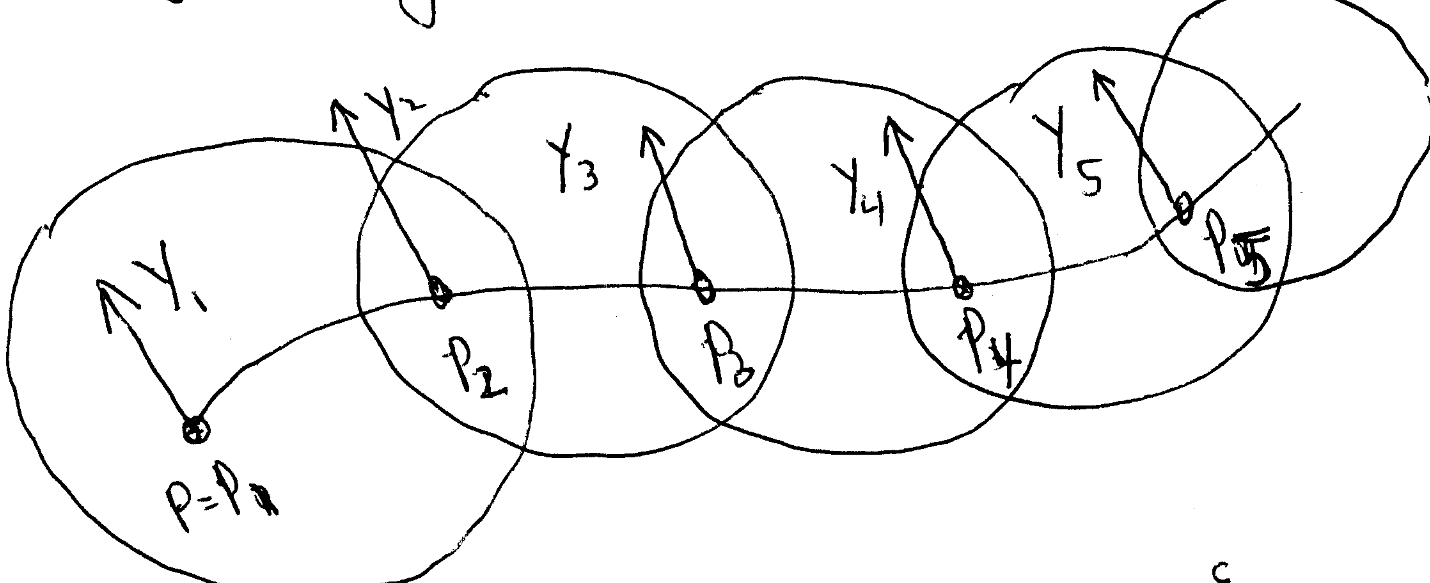
- **Assumption** (General Relativity) $\Gamma^i_{jk}(p) = 0$ in every locally inertial frame @ p:

$$\underset{\sim}{x} \text{ locally inertial: } \begin{cases} g_{ij}(p) = \eta_{ij} \\ g_{ij,k}(p) = 0 \end{cases}$$

$\Rightarrow$ Γ^i_{jk} for metric must be symmetric ($T^i_{jk} = 0$!)

- Conclude: In principle, given g, one can obtain the metric connection of GR by constructing a locally inertial frame @ p, & using (*) to obtain Γ^i_{jk} in any coord. syst.

- Turn this around: You can get ∥-trans. of Y by locally inertial frames:



- Cover $C(\xi)$ by n pts spaced $\overbrace{O(\tfrac{1}{n})}^{=\Delta\xi}$ apart
- ∥ translate Y^i as constant in each coord syst
- Transform Y^i to $(i+1)$'s coord system in overlap.
- Incure error $\Delta\xi^2$ in each $\Delta\xi$ step
- $Y^i_n = Y^i_1 + \sum_{i=1}^{n} \Delta\xi^2 \longleftarrow$ "$n\Delta\xi^2 = \frac{1}{\Delta\xi}\Delta\xi^2 = \Delta\xi \to 0$"

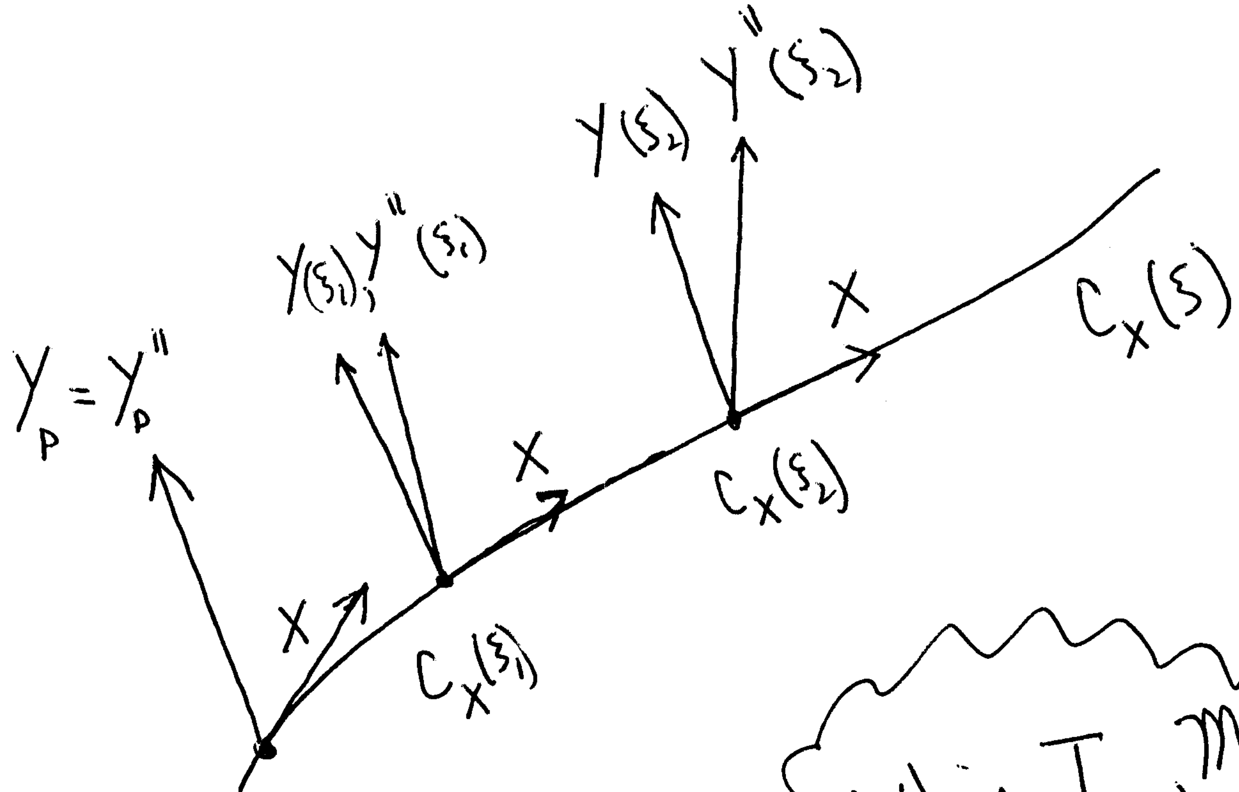
- Covariant Derivative: $\nabla_X Y$ defined by Γ
- Given 2 vector fields X, Y
- Let $C_X(\xi)$ be integral curve of X starting @ $C_X(0) = P$, so $\frac{dC_X(\xi)}{d\xi} = X_{C_X(\xi)}$
- Let $Y(\xi) = Y_{C_X(\xi)}$

$$Y_{\parallel}(\xi) \equiv \parallel\text{-trans of } Y_P \text{ to } C_X(\xi) \text{ along } C_X$$

- Defn: $$\nabla_X Y\Big|_P = \lim_{\xi \to 0} \frac{Y(\xi) - Y_{\parallel}(\xi)}{\xi}$$

Since both $Y(\xi)$ & $Y_{\parallel}(\xi)$ are coord indept, this gives coord indept notion of deriv of vector field Y in X direction.
(only depends on $X(P)$!)

Picture $\nabla_X Y$:



$c_X(0) = p$

both in $T_{c_X(\xi)}\mathcal{M}$

$$\left(\nabla_X Y\right)_p = \lim_{\xi \to 0} \frac{Y(\xi) - Y_{\parallel}(\xi)}{\xi}$$

- The covariant derivative corrects vector differentiation to a tensor operation: I.e.,

$$\nabla_X Y\Big|_p = \lim_{\xi \to 0} \frac{Y(\xi) - Y(0)}{\xi} + \lim_{\xi \to 0} \frac{Y(0) - Y_{\parallel}(\xi)}{\xi}$$

We only have a coord way to express this limit!

$$= X(Y) - \frac{dy^i}{d\xi}\frac{\partial}{\partial x^i}$$

$$= X(Y) + \Gamma^i_{jk}\, y^j X^k \frac{\partial}{\partial x^i}$$

coord indept but not a tensor

Γ gives us a coord expression for a coord indept thing

- In coordinates:

$$(\nabla_X Y)^i = \frac{dy^i}{d\xi} - \frac{dy^i_{\parallel}}{d\xi} = X^\sigma y^i_{,\sigma} + \underbrace{\Gamma^i_{jk}\, y^j X^k}_{\text{corrects } X(Y) \text{ to a tensor}}$$

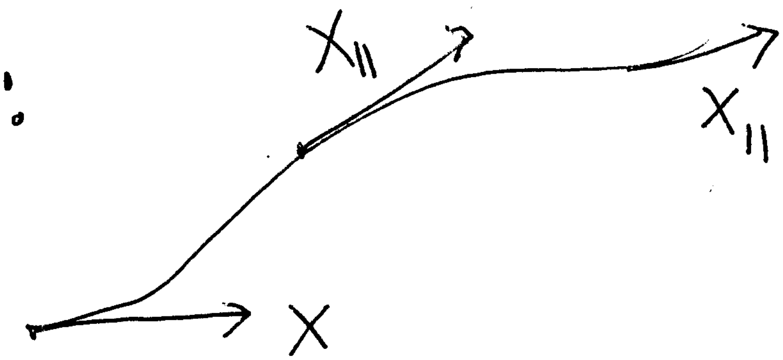
- **Conclude**: ∇ gives a coord indept expression to the Γ's

- **Defn**: Y is parallel along $c(\xi)$ if

$$\nabla_X Y = 0, \quad X = \frac{dc}{d\xi}$$

- **Defn**: a curve $\gamma(\xi)$ is a geodesic of Γ if $X = \frac{d\gamma}{d\xi}$ is parallel along γ.

Geodesic Equation:



$\nabla_X X = 0 \iff$

$$(\nabla_X X)^i = X^i \frac{\partial}{\partial x^i} X_{\gamma(\xi)} + \Gamma^i_{jk}(\gamma(\xi)) X^j X^k = 0$$

Since $\dot{\gamma}^i(\xi) = X^i \Rightarrow X^j \frac{\partial}{\partial x^j} \dot{\gamma}^i(\xi) = \ddot{\gamma}^i(\xi)$

$$\iff \boxed{\ddot{\gamma}^i + \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0}$$

• <u>Conclude</u>: ∇ gives a coordinate indept expression to the Γ's

<u>Properties</u>: ($X_i \in T_pM$, Y a <u>vector field</u>)

① $\nabla_{aX_1+bX_2} Y = a\nabla_{X_1} Y + b\nabla_{X_2} Y$ $\left\{\begin{array}{l}\text{any smooth fn's} \\ a,b : M \to \mathbb{R}\end{array}\right.$

② $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

③ $\nabla_X \left[f(p) Y\right] = f(p) \nabla_X Y + X(f) \nabla_X Y$

define $\nabla_X f = X(f)$ so Liebniz rule holds

④ $\nabla_X Y - \nabla_Y X = [X, Y] \equiv L_X Y$ (when $\Gamma^i_{jk} = \Gamma^i_{kj}$)

Pf ④: $\nabla_X Y - \nabla_Y X = X(Y) + \Gamma^i_{jk} Y^j X^k$

$$- Y(X) - \Gamma^i_{jk} X^j Y^k$$

$$= \overbrace{X(Y) - Y(X)}^{[X,Y]} + T^i_{jk} X^j Y^k$$

we assume Γ symmetric here on

◈ Extend ∇ to covectors ω by requiring:

$$(\nabla_X \omega)(Y) = \nabla_X(\omega(Y)) \quad \forall\, Y \text{ st } \nabla_X Y = 0$$

"so that $\nabla_X \omega = 0$ when $\omega(Y)$ evaluates parallel vector fields Y along $c_X(s)$ as constant."

<u>That</u> <u>is</u>: $\nabla_X \omega = (\nabla_X \omega)_{\dot\sigma}\, dx^{\dot\sigma}$

so $(\nabla_X \omega)(Y) = (\nabla_X \omega)_{\dot\sigma} Y^{\dot\sigma}$

subject to $(\nabla_X \omega)_\sigma Y^\sigma = \nabla_X(\omega(Y))$ when $\nabla_X Y = 0$.

So assume $\nabla_X Y = 0$, and calculate

$$\nabla_X \omega(Y) = X(\omega(Y)) = X^i \frac{\partial}{\partial x^i}(\omega_\sigma Y^\sigma)$$

$$= X^i\left(\frac{\partial}{\partial x^i}\omega_\sigma\right)Y^\sigma + X^i \omega_\sigma \frac{\partial}{\partial x^i}(Y^\sigma)$$

$$= \left(X^i \frac{\partial}{\partial x^i}\omega_\sigma\right)Y^\sigma + \omega_\sigma\left(\underbrace{\cancel{(\nabla_X Y)^\sigma}}_{0} - \Gamma^\sigma_{jk} Y^j X^k\right)$$

$$= \left(X^i \frac{\partial}{\partial x^i}\omega_\sigma - \Gamma^\tau_{\sigma k}\omega_\tau X^k\right)Y^\sigma$$

$$= (\nabla_X \omega)_\sigma Y^\sigma$$

$$\Rightarrow (\nabla_X \omega)_\sigma = X^i \omega_{\sigma,i} - \Gamma^\tau_{\sigma k}\omega_\tau X^k$$

We also write

$$\nabla_i Y = \nabla_{\frac{\partial}{\partial x^i}} Y = \left(Y^\sigma_{,i} + \Gamma^\sigma_{ij} Y^j\right)\frac{\partial}{\partial x^\sigma}$$

$$= Y^\sigma_{;i}\frac{\partial}{\partial x^\sigma}$$

$$\nabla_i \omega = \omega_{\sigma;i}\, dx^\sigma = \left(\omega_{\sigma,i} - \Gamma^\tau_{\sigma i}\omega_\tau\right)dx^\sigma$$

(10)

◊ we can extend ∇ to arb. tensor fields by asking:

$$[\nabla_X T](X_1, \dots, X_n, \omega^1, \dots, \omega^k)$$

$$= \nabla_X \underbrace{[T(X_1, \dots, X_k, \omega^1, \dots, \omega^k]}_{\text{scalar}}$$

for all $X_1 \cdots X_n$ $\omega^1 \cdots \omega^n$ // along $C_X(\xi)$.

<u>Formula</u>:

$\nabla_X (T^i_j \, dx^j \otimes \frac{\partial}{\partial x^i})$ has components

$$X^k \frac{\partial}{\partial x^k} T^i_j + \Gamma^i_{\sigma\tau} T^\sigma_j X^\tau - \Gamma^\sigma_{j\tau} T^i_\sigma X^\tau = (\nabla_X T)^i_j$$

a term for every contravariant indice

a term for every covariant indice.

(*)

DE

<u>Defn</u>: we let ∇T denote the (tensor) with components $T^{j_1 \cdots j_r}_{i_1 \cdots i_r ; j}$ when T has components $T^{j_1 \cdots j_r}_{i_1 \cdots i_r}$.

$$(\nabla X)^i_j = X^i_{;j} = X^i_{,j} + \Gamma^i_{\sigma j} X^\sigma \quad \text{etc.}$$

<u>Properties</u>:

① $\nabla_X T$ is a tensor for any tensor T.

② $\nabla_X (A \otimes B) = \nabla_X A \otimes B + A \otimes \nabla_X B$

③ $\nabla_X (T^i{}_i) = (\nabla_X T)^i{}_i$

Moregenerally, ∇_X commutes with contraction.

Ref MTW pg 223

- geodesics of a connection:

<u>Defn</u>: a geodesic is a curve along which the tangent vector is parallel.

<u>Equn</u> for geodesic of connection:

Let $x(\xi)$ be the x-coord curve $C_x(\xi)$, with tangent vector $\frac{dC_x}{d\xi} = X$

i.e.,
$$\frac{dx^i}{d\xi} = X^i.$$

then in x-coords, the condition $\nabla_X X = 0 \Rightarrow$

$$(\nabla_X X)^i = X^\sigma \frac{\partial}{\partial x^\sigma} X^i + \Gamma^i_{\sigma\tau} X^\sigma X^\tau$$

$$= \frac{d}{d\xi} X^i(\xi) + \Gamma^i_{\sigma\tau} X^\sigma X^\tau$$

$$\left.\begin{array}{l} = \boxed{\dfrac{d^2x^i}{d\xi^2} + \Gamma^i_{\sigma\tau}\dot{x}^i\dot{x}^j = 0} \\ \\ ! \text{ soln w i.c.} \quad x^i(0) = x^i_0 \\ \qquad\qquad\qquad \dot{x}^i(0) = \dot{x}^i_0 \end{array}\right\} \quad (**)$$

Note: if $x(\xi)$ solves (**) with initial vector X, then $x(c\xi)$ solves (**) with initial vector cX.

Thm: ① Two symmetric connections having the same geodesics agree. ② Geodesics for $\Gamma^i_{\alpha\beta}$ agree with geodesics for $\frac{1}{2}(\Gamma^i_{\alpha\beta}+\Gamma^i_{\beta\alpha})$.

Pf Ie. $\Gamma, \bar\Gamma$ two connections $\Rightarrow$ in x-coords,

$$\frac{d^2x^i}{d\xi^2} + \bar\Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \text{ iff } \frac{d^2x^i}{d\xi^2} + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$$

$$x^i(0) = x^i_0 \qquad \text{same i.c.'s.}$$
$$\dot{x}^i(0) = \dot{x}^i_0$$

choose: $\dot{x}^i_0 = \begin{cases} 1 & i=\alpha \\ 0 & \text{ow} \end{cases}$ fix $x^i_0 = x(p)$

Then at p:

$$\Rightarrow \frac{d^2x^i}{d\xi^2} = -\bar\Gamma^i_{\alpha\alpha}(\dot{x}^\alpha_0)^2 = -\Gamma^i_{\alpha\alpha}(\dot{x}^\alpha_0)^2$$

$$\Rightarrow \bar\Gamma^i_{\alpha\alpha} = \Gamma^i_{\alpha\alpha} \quad \text{all } \alpha$$

$$\dot{x}^i_0 = \begin{cases} 1 & i=\alpha,\ i=\beta \\ 0 & \text{ow} \end{cases}$$

$$\Rightarrow \frac{d^2x^i}{d\xi^2} = -\bar\Gamma^i_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - \bar\Gamma^i_{\beta\alpha}\dot{x}^\beta\dot{x}^\alpha = -\Gamma^i_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - \Gamma^i_{\beta\alpha}\dot{x}^\alpha\dot{x}^\beta$$

$$\Rightarrow \frac{1}{2}(\bar\Gamma^i_{\alpha\beta}+\bar\Gamma^i_{\beta\alpha}) = \frac{1}{2}(\Gamma^i_{\alpha\beta}+\Gamma^i_{\beta\alpha})$$

<u>Conclude</u>: Symmetry $\Rightarrow \Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$

Non-symmetry $\Rightarrow$ geodesics for Γ are the same as the geodesics for the symmetric connection $\tilde{\Gamma}^i_{\alpha\beta} = \frac{1}{2}\left(\Gamma^i_{\alpha\beta} + \Gamma^i_{\beta\alpha}\right)$ ✓

~~Assume Γ is the connection for metric g.~~

~~In a local Lorentz frame at p, the geodesics for g agree with the geodesics for Γ, $\Gamma(p) = 0$, to 1st order.~~

~~their geodesics to agree:~~

~~<u>Theorem</u>:~~

<u>Theorem</u>: (The connection that goes with metric g)

Assume g is a Lorentzian metric with signature $\eta = \mathrm{diag}(-1,1,1,1)$. Let Γ be a connection whose components vanish in a coordinate frame x where (cond'ts for local inertial frame)

$$g_{ij}(p) = \eta \qquad (i)$$

$$g_{ij,h}(p) = 0. \qquad (ii)$$

Then in any other coord. frame y we must have

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma}\left\{-g_{\alpha\beta,\sigma} + g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha}\right\}. \qquad (iii)$$

<u>thus</u> giving a formula for the Christoffel symols for a **unique** <u>symmetric</u> connection determined by the metric g.

<u>Proof</u>: The transformation law for a connection is given by

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{i}_{jk} \frac{\partial y^{\gamma}}{\partial x^{i}} \frac{\partial x^{j}}{\partial y^{\alpha}} \frac{\partial x^{k}}{\partial y^{\beta}} + \frac{\partial^2 x^{i}}{\partial y^{\alpha} \partial y^{\beta}} \frac{\partial y^{\gamma}}{\partial x^{i}} . \qquad (iv)$$

Since $\Gamma^{i}_{jk} = 0$ at p, we have

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} \equiv \Gamma^{\gamma}_{\alpha\beta}(p) = \frac{\partial^2 x^{i}}{\partial y^{\alpha} \partial y^{\beta}} \frac{\partial y^{\gamma}}{\partial x^{i}} . \qquad (v)$$

Now the components of the metric $g_{\alpha\beta}$ are given by

$$g_{\alpha\beta} = \frac{\partial x^{i}}{\partial y^{\alpha}} \eta_{ij} \frac{\partial x^{j}}{\partial y^{\beta}} \qquad (vi)$$

$$\Leftrightarrow \quad g = J^{t} \eta J \qquad \text{(matrix notation)} \quad (vii)$$

Differentiating (vi) and using (ii) gives

$$g_{\alpha\beta,\gamma}(p) = \underbrace{\eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\gamma} \frac{\partial x^j}{\partial y^\beta}}_{\Delta_{\alpha\gamma,\beta}} + \underbrace{\eta_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial^2 x^j}{\partial y^\beta \partial y^\gamma}}_{\Delta_{\beta\gamma,\alpha}}.$$

Now

Notation

So $\quad \Delta_{\alpha\gamma,\beta} = \Delta_{\gamma\alpha,\beta}$. (equality of mixed partials)

Therefore,

$$- g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} + g_{\beta\gamma,\alpha} = - \cancel{\Delta_{\alpha\gamma,\beta}} - \cancel{\Delta_{\beta\gamma,\alpha}} + \cancel{\Delta_{\gamma\beta,\alpha}} + \Delta_{\alpha\beta,\gamma} + \Delta_{\beta\alpha,\gamma} + \cancel{\Delta_{\gamma\alpha,\beta}}$$

$$= 2\Delta_{\alpha\beta,\gamma} = 2\eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^j}{\partial y^\gamma}$$

Now consider:

$$\left(g^{\alpha\beta}\right)_{4\times 4} \equiv \left(g_{\alpha\beta}\right)^{-1}_{4\times 4} = \left(J^t \eta J\right)^{-1} = J^{-1}\eta\left(J^t\right)^{-1}$$

Consider now the expression

$$\frac{1}{2} g^{\gamma\sigma}\left\{-g_{\alpha\beta,\sigma} + g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha}\right\}$$

$$= \underbrace{g^{\gamma\sigma}}_{J^{-1}\eta\left(J^t\right)^{-1}} \underbrace{\left\{\eta_{ij} \frac{\partial^2 x^j}{\partial y^\alpha \partial y^\beta} \frac{\partial x^i}{\partial y^\sigma}\right\}}_{J^{tr}\eta M_\beta}$$

$$M_\beta = \frac{\partial^2 x^j}{\partial y^\alpha \partial y^\beta}$$

$$= J^{-1}\cancel{\eta}\cancel{\left(J^t\right)^{-1}}\cancel{J^{tr}}\cancel{\eta} M_\beta = J^{-1} M_\beta$$

$$= \frac{\partial y^\gamma}{\partial x^i} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} = \Gamma^\gamma_{\alpha\beta}$$

Cor ①: The geodesics for metric g satisfy the geodesic equation, Γ given in (iii):

$$\frac{d^2x^i}{d\xi^2} + \Gamma^i_{jk}\frac{dx^j}{d\xi}\frac{dx^k}{d\xi} = 0. \qquad \text{(GD)}$$

Thm: curves that satisfy (GD) are critical points of the $(\text{length})^2$ (Do timelike & spacelike curves separately)

$$\text{Length} \equiv A[c(\cdot)] = \int_{\xi_1}^{\xi_2} \|\dot{x}(\xi)\|^2 \, d\xi$$

$$= \int_{\xi_1}^{\xi_2} \langle \dot{x}, \dot{x} \rangle \, d\xi = \int_{\xi_1}^{\xi_2} g_{ij}\dot{x}^i\dot{x}^j \, d\xi$$

$\Rightarrow$ The Euler-Lagrange eqns are equiv. to (GD) Ref Spivak / Dubrovin Novikov Modern Geom I

Cor (2). Let $T_{\alpha\beta}^{\gamma}$ be any torsion tensor,

" $T_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma} - \Gamma_{\beta\alpha}^{\gamma}$ ",

so $T_{\alpha\beta}^{\gamma}$ any antisymmetric tensor. Then if $\Gamma_{\alpha\beta}^{\gamma}$ is given by (iii), it follows that

$$\overline{\Gamma}_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma} + T_{\alpha\beta}^{\gamma}$$

has the same geodesics as Γ, but is not symmetric.

We could show: non-symmetry introduces a twist rel to nearby geodesics.

Cor (3) $\exists!$ symmetric connection whose geodesics are geodesics of the metric.

<u>Lemma</u>: Let g have connection $\Gamma \Leftrightarrow \nabla$.

Then

① $\nabla_z g = 0 \quad \forall z$

~~and~~

$\cancel{② \ \nabla_z (g_{ij} X^i Y^j) = g_{ij} (\nabla_z X)^i Y^j + g_{ij} X^i (\nabla_z Y)^j}$

~~Note: ② ⇒ angles & lengths are preserved under ‖ translation.~~

<u>Proof</u>: For ①, go to a locally Lorentzian coord frame where $g_{ij,k} = 0$ & $\Gamma^i_{jk} = 0$.

Formula for cov. deriv $\Rightarrow \nabla_z g = 0$.

$\nabla_z g$ a <u>tensor</u> $\Rightarrow \nabla_z g = 0$ in all coords. ✓

$\cancel{\text{For ②, write } \nabla_z (g \otimes X \otimes Y)}$

<u>Corollary</u>: For metric connection: ∇ for g:

② $\nabla_Z \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$

③ $\nabla_Z \left(g_{\sigma i} X^\sigma dx^i \right)_i = g_{\tau i} \nabla_Z \left(X^\sigma \frac{\partial}{\partial x^\sigma} \right)^\tau$

More generally - ∇_Z commutes with raising & lowering of indices (FIP)

<u>Note</u>:

② $\Rightarrow$ if X, Y are parallel along $C_X(s)$, then $\nabla_Z \langle X, Y \rangle = Z(\langle X, Y \rangle) = 0 \Rightarrow$

$\parallel$-translation preserves lengths & angles betw vectors.

Proof

For ②, write $T = g \otimes X \otimes Y$. Then

$$\nabla_Z T = \underbrace{\nabla_Z g}_{0} \otimes X \otimes Y + g \otimes \nabla_Z X \otimes Y + g \otimes X \otimes \nabla_Z Y$$

But: ∇_Z (a contraction of T) = contraction of $(\nabla_Z T)$

LHS: $$\nabla_Z \big(\underbrace{g_{ij} X^k Y^\ell \, dx^i \otimes dx^j \otimes \tfrac{\partial}{\partial x^k} \otimes \tfrac{\partial}{\partial x^\ell}}_{\text{contract twice}} \big)$$

$$\Rightarrow \nabla_Z \left(g_{ij} X^i Y^j \right) = \nabla_Z \langle X, Y \rangle$$

RHS: $$\underbrace{g_{ij} (\nabla_Z X)^k Y^\ell \, dx^i \otimes dx^j \otimes \tfrac{\partial}{\partial x^k} \otimes \tfrac{\partial}{\partial x^\ell}}_{\text{contract twice}}$$

$$\Rightarrow g_{ij} (\nabla_Z X)^i Y^j = \langle \nabla_Z X, Y \rangle$$

Sim: contract: $g \otimes X \otimes \nabla_Z Y \to \langle X, \nabla_Z Y \rangle$

<u>Corollary</u>: along a geodesic, $\langle X, X \rangle$ is constant (i.e., $\nabla_X X = 0$) $\Rightarrow$

timelike geodesics remain timelike

Spacelike " " spacelike

Null " " null. ✓

<u>Assmt</u>: Free fall paths are timelike geodesics

light <u>rays</u> follow null geodesics.

Lie Deriv as a Covariant Derivative

<u>Theorem</u>: Let Γ be any connection & ∇ its covariant derivative. Then

$$(L_Y T)^i_j \equiv \left(\frac{d}{ds} \varphi_{s*} T_{\varphi_{-s}(p)} \Big|_{s=0} \right)^i_j$$

$$= (\nabla_Y T)^i_j - T^\sigma_j (\nabla_\sigma Y)^i + T^i_\sigma (\nabla_j Y)^\sigma \quad (*)$$

one for every up index

one for every down index

Cor

<u>Cor</u>: $$(L_Y g)_{ij} = \underbrace{\cancel{(\nabla_Y g)_{ij}}}_{0} + g_{\sigma j} \nabla_i Y^\sigma + g_{i\sigma} \nabla_j Y^\sigma$$

$$= \nabla_i Y_j + \nabla_j Y_i \equiv Y_{j;i} + Y_{i;j}$$

P.f. of theorem: $L_Y f = Y(f)$; $L_Y X = [Y, X]$

- $L_Y(W_\sigma X^\sigma) = Y(W_\sigma X^\sigma)$ ← apply to every down index $T_{\cdots\sigma\cdots}$

Also

$$L_Y(W_\sigma X^\sigma) = (L_Y W)_\sigma X^\sigma + W_\sigma (L_Y X)^\sigma$$

$$= (L_Y W)_\sigma X^\sigma + W_\sigma [Y, X]^\sigma \qquad (A)$$

$$= (L_Y W)_\sigma X^\sigma + W_\sigma \left((\nabla_Y X)^\sigma - (\nabla_X Y)^\sigma \right)$$

$$Y(W_\sigma X^\sigma) = (\nabla_Y W)_\sigma X^\sigma + W_\sigma (\nabla_Y X)^\sigma \qquad (B)$$

Equating (A) & (B):

$$(\nabla_Y W)_\sigma X^\sigma = (L_Y W)_\sigma X^\sigma + W_\sigma (\nabla_X Y)^\sigma$$

$X = \frac{\partial}{\partial x^i}$: $\quad ({}_Y W)_i = (\nabla_Y W)_i + W_\sigma \nabla_{\frac{\partial}{\partial x^i}} Y^\sigma$

- <u>Divergence</u>: Given a vector field X, define

$$\operatorname{div} X = X^i{}_{;i} \quad \text{a scalar.}$$

- in Local Lorentz coord's, $\operatorname{div} X = X^i{}_{,i}$ = classical divergence.

<u>Theorem</u>: $\operatorname{div} X = \frac{1}{\sqrt{-g}}\left(X^i \sqrt{-g}\right)_{,i}$

<u>Proof</u>: $X^i{}_{;i} = X^i{}_{,i} + \Gamma^i_{\sigma i} X^\sigma$

We evaluate $\Gamma^i_{\sigma i} = \Gamma^i_{i\sigma}$:

We know from $\nabla g = 0$ that

$$0 = g_{ij,k} - \Gamma^\sigma_{ik} g_{\sigma j} - \Gamma^\sigma_{jk} g_{\sigma i}$$

mult by g^{ij} & contract:

$$0 = g^{ij} g_{ij,k} - \Gamma^i_{ik} - \Gamma^j_{jk} \Rightarrow \Gamma^i_{ik} = \frac{1}{2} g^{ij} g_{ij,k}$$

i.e., $g_{\sigma j} g^{ij} = \delta^i_\sigma$ ✓

But: $g^{ij} = \frac{1}{g}\Delta^{ij}$ ← Matrix of cofactors $g = \det g_{ij}$ $\Delta^{ij} = (-1)^{i+j}(\underset{ij}{\text{subdet}}\, g_{ij})$

$$\Delta^{ij} = \frac{\partial g}{\partial g_{ij}}$$ ← just expand g by jth colm.

$$\Rightarrow \Gamma^i_{ik} = \frac{1}{2} g^{ij} g_{ij,k} = \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2}\frac{\partial}{\partial x^k}\ln|g|$$

$$= \frac{\partial}{\partial x^k}\log\sqrt{-g} = \underbrace{\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g}}$$

Thus: $$\boxed{\Gamma^i_{ik} = \frac{\partial}{\partial x^k}\log\sqrt{-g}},$$

and

$$\operatorname{div} X = X^i{}_{,i} + \overbrace{\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial x^k}\sqrt{-g}\right)}X^k$$

$$= \frac{1}{\sqrt{-g}}\left(X^i\sqrt{-g}\right)_{,i}$$

Gauss Theorem: A coordinate indept SKIP version of div theorem that holds for vector fields:

Theorem: $F = F^i \frac{\partial}{\partial x^i}$ a vector field. Then

$$\int_{\Omega^4} \operatorname{div} F \, \eta = \int_{\partial\Omega^4} \frac{\langle F, n\rangle}{\langle n, n\rangle} \eta \lrcorner n = \int_{\partial\Omega^4} F^\alpha d\Sigma_\alpha$$

Here: $\eta \equiv \sqrt{-g}\,\varepsilon$ volume form

$\Omega \equiv$ 4-dimensional region

$\operatorname{div} F \equiv F^\sigma{}_{;\sigma}$ a scalar

$n \equiv$ unit outward normal to $\partial\Omega$, normal relative to metric g, $\langle\,\rangle$.

$$d\Sigma_\alpha = \eta \lrcorner \frac{\partial}{\partial x^\alpha}$$

<u>Proof</u>: By previous result,

$$F^{\sigma}{}_{;\sigma} = \frac{1}{\sqrt{-g}}\left(F\sqrt{-g}\right)_{,\sigma}.$$

Assume Ω lies in a single coord chart: x:
Then the Classical Divergence Thm holds as follows:

$$\int_{\Omega} F^{\sigma}{}_{;\sigma}\,\eta = \int_{\Omega} F^{\sigma}{}_{;\sigma}\sqrt{-g}\,\mathcal{E} = \int_{\Omega}\frac{1}{\sqrt{-g}}\left(F^{\sigma}\sqrt{-g}\right)_{,\sigma}\sqrt{-g}\,\mathcal{E}$$

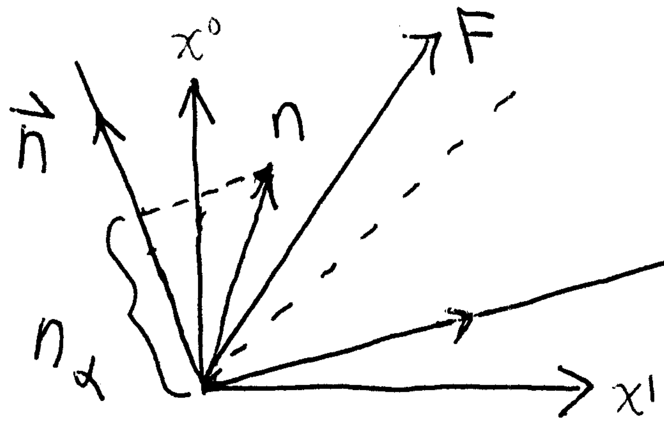
$$= \int_{\Omega}\left(F^{\sigma}\sqrt{-g}\right)_{,\sigma}\underset{\substack{\uparrow\\ \text{coordinate}\\ \text{volume } dV}}{\mathcal{E}} \underset{\substack{\uparrow\\ \text{Classical}\\ \text{Divergence}\\ \text{Thm}}}{=} \int_{\partial\Omega} F\cdot\vec{n}\,\sqrt{-g}\,\underbrace{\mathcal{E}\lrcorner\,\vec{n}}_{\substack{dS\\ \equiv\text{ Euclidean}\\ \text{surface area}}} = *$$

where $\vec{n}$ is the <u>coordinate</u> unit outer normal to $\partial\Omega$.

<u>Now</u>: if $\vec{n}$ is the <u>Euclidean</u> <u>coordinate unit normal</u>,

and $n \equiv n^\alpha \frac{\partial}{\partial n^\alpha}$ is the metric unit normal,

then $\vec{n}^i \neq n_i$: Eg



$$* = \int_{\partial\Omega} \sqrt{-g}\, \varepsilon \lrcorner (F\cdot\vec{n})\vec{n} = \int_{\partial\Omega} \sqrt{-g}\, \varepsilon \lrcorner \Big\{ (F\cdot\vec{n})\vec{n} + \underbrace{F-(F\cdot\vec{n})\vec{n}}_{\text{in } T_p\partial\Omega} \Big\}$$

no contribution for $X_1 \cdots X_3 \in T\partial\Omega$

$$= \int_{\partial\Omega} \eta \lrcorner F = \int_{\partial\Omega} F^\alpha d\Sigma_\alpha$$

$$d\Sigma_\alpha = \eta \lrcorner \frac{\partial}{\partial x^\alpha}$$

But we can also write

$$\int_{\partial\Omega} \eta \lrcorner F = \int_{\partial\Omega} \eta \lrcorner \Big\{ \frac{\langle F, n\rangle}{\langle n, n\rangle} n + \underbrace{F - \frac{\langle F, n\rangle}{\langle n, n\rangle} n}_{\text{in } T\partial\Omega} \Big\}$$

$$= \int_{\partial\Omega} \eta \lrcorner \Big\{ \frac{\langle F, n\rangle}{\langle n, n\rangle} n \Big\} = \int_{\partial\Omega} \underbrace{\frac{\langle F, n\rangle}{\langle n, n\rangle}}_{\substack{\text{metric}\\ \text{version of}\\ \text{“}F\cdot n\text{”}}} \underbrace{\eta \lrcorner n}_{\substack{\text{metric}\\ \text{version}\\ \text{of}\\ \text{“}dS\text{”}}}$$

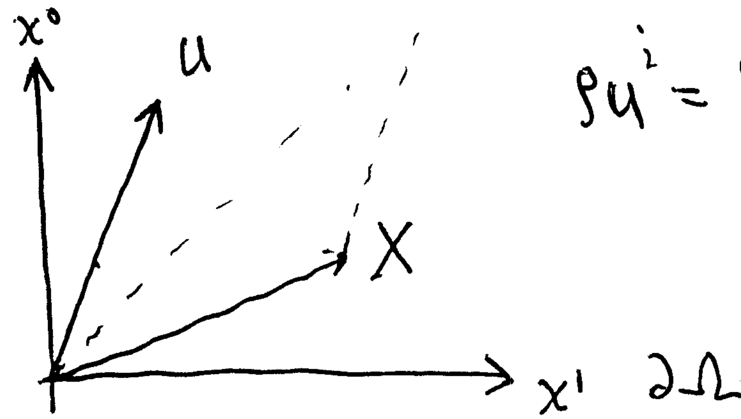
Application: Cons. of mass principle

Ex: $\rho(x) \equiv \frac{\text{mass}}{\text{3-vol}}$ in rest frame $\approx$ dust.

$u = \frac{dx^i}{d\xi}$ 4-velocity of particles

$$d\xi = \sqrt{(dx^0)^2 - |d\underset{\sim}{x}|^2} = \sqrt{1-|\underset{\sim}{v}|^2}\, dx^0$$

$$\frac{dx^0}{d\xi} = \frac{1}{\sqrt{1-|\underset{\sim}{v}|^2}} \equiv \text{Lorentz contraction Factor}$$



$$\rho u^i = \rho \frac{dx^0}{d\xi}\frac{dx^i}{dx^0}$$

$$\frac{\text{mass}}{\text{3-vol}} \text{ in } x\text{-frame} = \rho\frac{dx^0}{d\xi} = \left\langle \rho u, \frac{\partial}{\partial x^0}\right\rangle$$

$$\frac{\text{mass}}{\text{area } \Delta x^0} \text{ in } x\text{-frame} = \underbrace{\rho\frac{dx^0}{d\xi}}_{\frac{\text{mass}}{\text{3-vol}}}\underbrace{\frac{dx^1}{dx^0}}_{\frac{\text{dist}}{\text{time}}} = \left\langle \rho u, \frac{\partial}{\partial x^1}\right\rangle$$

<u>Conclude</u>: At each pt. on $\partial\Omega$

$$\langle \rho u, n \rangle = \frac{\text{mass}}{\text{metric 3-vol}} \equiv \begin{cases} \text{density if } n \text{ timelike} \\ \text{mass flux if } n \text{ spacelike} \end{cases}$$

coordinate indep't meaning

$\therefore \quad \dfrac{\langle \rho u, n\rangle}{\langle n, n\rangle}\, \eta \lrcorner n \; (X_1, X_2, X_3)$

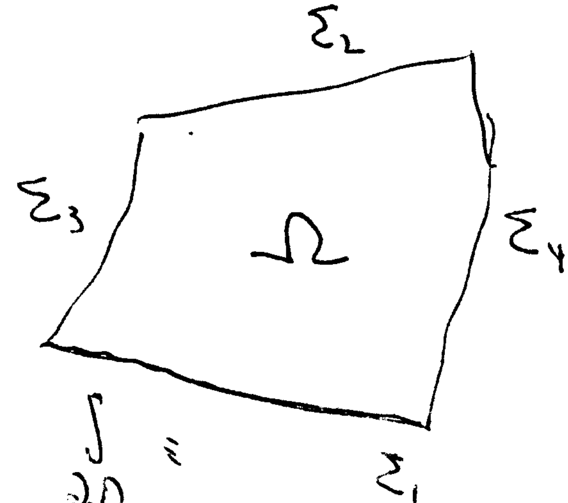
evaluates the total mass in 3-llogram $X_1\, X_2\, X_3$ (per coord vol)

$$\int_{\partial\Omega} \frac{\langle \rho u, n\rangle}{\langle n, n\rangle}\, \eta \lrcorner n \equiv \int_{\partial\Omega_x} \frac{\langle \rho u, n\rangle}{\langle n, n\rangle}\, \eta \lrcorner n \left(\frac{\partial}{\partial x^1}\, \frac{\partial}{\partial x^2}\, \frac{\partial}{\partial x^3}\right) dx^1 dx^2 dx^3$$

$\int_{\Sigma_1}$ = "total mass at Σ_1"

$\int_{\Sigma_2}$ = "total mass at Σ_2"

$\int_{\Sigma_3, \Sigma_4}$ = "total mass passing out of Ω through $\Sigma_3\ \Sigma_4$"



$\int_{\partial\Omega_x}$ =

$\Rightarrow$ Global Coord indept statement of Cons. of mass is

$$0 = \int_{\partial\Omega} \frac{\langle \rho u, n\rangle}{\langle n, n\rangle}\, \eta \lrcorner n = \int_{\Omega} \operatorname{div} \rho u \quad \Leftrightarrow \quad \boxed{\operatorname{div} \rho u = 0}$$

• <u>Note</u>: since $\displaystyle\int_{\partial\Omega} \frac{\langle \rho u, n\rangle}{\langle n, n\rangle}\, \eta \lrcorner n = \int_{\partial\Omega} \rho u^i \, d\Sigma_i$,

we can think of $\rho u^i d\Sigma_i$ as "eating 3-vol's $X_1 \wedge X_2 \wedge X_3$ & evaluating the total mass in them $\equiv$ mass flux if one of the X_i is timelike'

Ref MTW.

For energy & momentum:

$\rho \frac{dx^0}{d\xi} u$ energy flux vector

$\rho \frac{dx^i}{d\xi} u$ i-momentum flux vector

extra factor of $\frac{dx^0}{d\xi}$ needed because $\frac{\text{energy}}{\text{vol}}$ increases with velocity due to Lorentz Contr effect + increase in K.E.

Components of tensor: $T^{ij} = \rho u^i u^j$

<u>Conclude</u>: $F = \rho \frac{dx^0}{d\xi} u$ satis Global Cons Law

$$\int_{\partial\Omega} \frac{\langle F, n\rangle}{\langle n, n\rangle} \eta \lrcorner n = 0 \qquad (*)$$

<u>only</u> so long as F transforms like a vector. It <u>does</u> under Linear coord

transformations $\Rightarrow$ global inv. cons. law holds in Special Relativity when we restrict to ~~exact~~ Lorentz transformations.

In general relativity, we can only require local cons of energy

$$\boxed{\operatorname{div} T = 0 = T_{i\ ;\sigma}^{\ \sigma}}$$

This reproduces (*) in flat space & approximately reproduces (*) in Loc Lorentz frames where $g_{ij} = \eta_{ij} + O(|x|^2)$.